J Elasticity (2006) 85: 45–63
DOI 10.1007/s10659-006-9070-4

On Poisson's Ratio in Linearly Viscoelastic Solids

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Received: 19 April 2005 / Accepted: 20 April 2006 /

Abstract Poisson's ratio in viscoelastic solids is in general a time dependent (in the time domain) or a complex frequency dependent quantity (in the frequency domain). We show that the viscoelastic Poisson's ratio has a different time dependence depending on the test modality chosen; interrelations are developed between Poisson's ratios in creep and relaxation. The difference, for a moderate degree of viscoelasticity, is minor. Correspondence principles are derived for the Poisson's ratio in transient and dynamic contexts. The viscoelastic Poisson's ratio need not increase with time, and it need not be monotonic with time. Examples are given of material microstructures which give rise to designed time dependent Poisson's ratios.

Mathematics Subject Classifications (2000) P21018 · C32070 · T15001

Key words Poisson's ratio · linearly viscoelastic solids · correspondence principle · time dependence · frequencey dependence

1. Introduction

Does a stretched viscoelastic rod get fatter or thinner with time [1]? The transverse deformation of such a rod is described by the Poisson's ratio ν, which in viscoelastic materials depends on time or on frequency. Does it matter whether the rod is held

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at constant extension in relaxation, or whether it is held at constant axial stress in creep?

As in the case of elastic solids, the Poisson's ratio in linear viscoelasticity is used in the calculation of stress and strain distributions when these are expressed in terms of a modulus and a Poisson's ratio. In particular, three-dimensional stress fields such as those associated with stress concentration depend on Poisson's ratio. For example, stress in the vicinity of a bonded joint between dissimilar materials is sensitive to Poisson's ratio [2]. In a viscoelastic material, a time-dependent Poisson's ratio will be associated with time-dependent stress and deformation, so stress concentration factors and interface stresses can depend on time and frequency.

As for scientific applications, in isotropic materials, one can infer the bulk modulus from the Poisson's ratio and the shear or Young's modulus. Such inference is of interest since it is difficult to measure the viscoelastic bulk modulus [3], however owing to the nature of the interrelation equations, high precision is required in the input measurements [4]. Moreover one must achieve confidence in the isotropy of the specimen for such inference to be meaningful.

In viscoelastic solids, Poisson's ratio may be defined in several ways. Several authors have expressed concern about some definitions [5] of Poisson's ratio. For example, it is meaningful to consider Poisson's ratio as the ratio of time-dependent transverse to longitudinal strain in axial extension, provided one recognizes the distinction between creep and relaxation. Poisson's ratio defined as a ratio of Fourier transforms does not have a straightforward physical interpretation. A time-independent Poisson's ratio which may be assumed for simplicity of calculation is inconsistent with experimental data for most materials and requires unrealistic theoretical constructs.

As for experiment, one can determine Poisson's ratio directly from measured axial and transverse strains, or, in isotropic solids, infer it from time-dependent Young's and shear moduli. Experimental inferences of Poisson's ratio from direct or indirect data requires high accuracy as well as care in the design of experiments. It is difficult to directly measure the viscoelastic bulk modulus, therefore it is of interest to infer bulk properties from the axial modulus and Poisson's ratio, which are easier to obtain. In that vein, Tschoegl et al. [6] also adduce a reference to Lu et al. [7] in which Poisson's ratio must be determined to four significant digits to infer the bulk modulus. Inference of the bulk modulus from shear and axial modulus measurements requires high precision in the input data. In polymers as well as in other materials, viscoelastic properties depend on temperature, details of specimen preparation, aging time after preparation, as well as time/frequency. Therefore the input viscoelastic functions should be measured upon the same specimen, in the same environment, at the same time, and with high accuracy and precision. In stiff materials such as metals, strain can be measured using bonded strain gages. In compliant materials such as elastomers, optical methods [8] such as laser shadow casting, speckle interferometry, or moiré are appropriate. It has been suggested [6] that the time dependent Poisson's ratio $\nu(t)$ must be monotonically non-decreasing in all cases and that experimental results which indicate otherwise must be erroneous by virtue of the theory of viscoelasticity.

In the present work, the viscoelastic Poisson's ratio is shown to have a different time dependence for various test modalities. No assumptions are made specific to polymers. Examples are given of materials with decreasing and non-monotonic time dependent Poisson's ratio.

2. Viscoelastic Poisson's Ratio in Different Modalities

The viscoelastic Poisson's ratio is here calculated in several modalities which are amenable to experiment. The correspondence principle is applied to the moduli but not to the Poisson's ratio. These are defined in terms of the longitudinal and transverse strains in linear viscoelastic materials undergoing uniaxial tension. Expressions for these strains are developed through application of the correspondence principle. This is a tool that allows one to use a solution to an elasticity problem to obtain the solution to the corresponding viscoelasticity problem (with the same geometry and boundary conditions). In order to apply the correspondence principle, let $A(t)$ be a physical quantity such as strain and $B(t)$ be a material property. Let $\underline{\mathrm{A}}(p)$ and $\underline{\mathrm{B}}(p)$ be their Laplace transforms, with p as the Laplace transform variable. In an expression in elasticity, A is replaced by $\underline{\mathrm{A}}(p)$ and B is replaced by $p\underline{\mathrm{B}}(p)$. The solution for time-dependent behavior of a viscoelastic material is then obtained by applying an inverse Laplace transform.

2.1. Relaxation in Tension

Consider uniaxial tension in an elastic bar having elastic modulus E and Poisson's ratio ν. The longitudinal strain ε_{L} is

$$\varepsilon_{\mathrm{L}} = \sigma_{\mathrm{L}} \big/ E. \tag{1}$$

The transverse strain $\varepsilon_{\mathrm{tr}}$ is, in terms of Poisson's ratio ν,

$$\varepsilon_{\mathrm{tr}} = -\nu\varepsilon_{\mathrm{L}} = -\nu\sigma_{\mathrm{L}} \big/ E. \tag{2}$$

The Poisson's ratio ν is written [9] as follows for an isotropic elastic solid in terms of the bulk modulus B or the bulk compliance $\kappa = 1/B$:

$$\nu = \frac{1}{2} - \frac{\mathrm{E}}{6\mathrm{B}} = \frac{1}{2} - \frac{1}{6}\kappa\mathrm{E}. \tag{3}$$

By (2) and (3), the transverse strain can be written in the form,

$$\varepsilon_{\mathrm{tr}} = -\left[\frac{1}{2} - \frac{1}{6}\kappa E\right]\varepsilon_{\mathrm{L}}. \tag{4}$$

This relation between strains and moduli for elastic materials can be converted to one for viscoelastic materials by use of the correspondence principle. Assume first for simplicity of analysis that the bulk compliance is constant in time. The assumption of a constant bulk compliance is a good approximation for polymers in the glass–rubber transition, but as will be seen below does not apply in general. Applying the correspondence principle to (4) gives

$$\underline{\varepsilon}_{\mathrm{tr}}(p) = -\left[\frac{1}{2} - \frac{1}{6}\kappa p\underline{E}(p)\right]\underline{\varepsilon}_{\mathrm{L}}(p). \tag{5}$$

Transforming back to the time domain via the convolution theorem and making use of the derivative theorem of the Laplace transform, we obtain the following solution for the transverse strain in a viscoelastic material:

$$\varepsilon_{\mathrm{tr}}(t) = -\frac{1}{2}\varepsilon_{\mathrm{L}}(t) + \frac{1}{6}\kappa\int_0^{\mathrm{t}} E(t-\tau)\frac{\mathrm{d}\varepsilon_{\mathrm{L}}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{6}$$

Equation (6) gives the transverse strain for any history of longitudinal strain in terms of the (constant) bulk modulus and the relaxation function in tension. In stress relaxation, the longitudinal strain is $\varepsilon_{\mathrm{L}}(t) = \varepsilon_0 H(t)$ with $H(t)$ as a Heaviside step function of time t. The step function can be chosen to begin an infinitesimal time after zero to avoid a surface term. Substituting in (6) gives

$$\nu_{\mathrm{r}}(t) = -\frac{\varepsilon_{\mathrm{tr}}(t)}{\varepsilon_0} = \frac{1}{2} - \frac{1}{6}\kappa E(t). \tag{7}$$

This ratio of transverse to longitudinal strains is a Poisson's ratio, so we call it $\nu_{\mathrm{r}}(t)$, a Poisson's ratio in relaxation. Observe that $\nu_{\mathrm{r}}(t)$ is less time-dependent than $E(t)$ since for a typical value $\nu = 0.3$, the second (time varying) term in (7) is about half the total.

If bulk relaxation is allowed, (5) becomes

$$\underline{\varepsilon}_{\mathrm{tr}}(p) = -\left[\frac{1}{2} - \frac{1}{6}p\underline{\kappa}(p)p\underline{E}(p)\right]\underline{\varepsilon}_{\mathrm{L}}(p). \tag{8}$$

In contrast to (5), (8) contains a product of three functions on the right side. To facilitate inverse transformation via the convolution theorem (in which the Laplace transform of a convolution of two functions is a product of the Laplace transforms), define

$$\underline{E}(p)p^2\underline{\varepsilon}_{\mathrm{L}}(p) = \underline{Q}(p). \tag{9}$$

Then (8) becomes

$$\underline{\varepsilon}_{\mathrm{tr}}(p) = -\frac{1}{2}\underline{\varepsilon}_{\mathrm{L}}(p) + \frac{1}{6}\underline{\kappa}(p)\underline{Q}(p). \tag{10}$$

Taking an inverse transformation, as is done in using the correspondence principle, we obtain

$$\varepsilon_{\mathrm{tr}}(t) = -\frac{1}{2}\varepsilon_{\mathrm{L}}(t) + \frac{1}{6}\int_0^{\mathrm{t}} \kappa(t-\tau)Q(\tau)\mathrm{d}\tau. \tag{11}$$

Next, $Q(\tau)$ is determined from Equation (9) by inverse transformation. Substituting (12) into (11), we obtain the following:

$$Q(\tau) = \int_0^{\tau} E(\tau-\eta)\frac{\mathrm{d}^2\varepsilon_{\mathrm{L}}(\eta)}{\mathrm{d}\eta^2}\mathrm{d}\eta. \tag{12}$$

Substituting (12) into (11), we obtain the following:

$$\varepsilon_{\mathrm{tr}}(t) = -\frac{1}{2}\varepsilon_{\mathrm{L}}(t) + \frac{1}{6}\int_0^{\mathrm{t}} \kappa(t-\tau)\int_0^{\tau} E(\tau-\eta)\frac{\mathrm{d}^2\varepsilon_{\mathrm{L}}(\eta)}{\mathrm{d}\eta^2}\mathrm{d}\eta\mathrm{d}\tau. \tag{13}$$

As done following (6), let $\varepsilon_{\mathrm{L}}(t) = \varepsilon_0 H(t)$ in (13). Then, a more general expression for Poisson's ratio in stress relaxation in (7) is,

$$\nu_{\mathrm{r}}(t) = \frac{1}{2} - \frac{1}{6}\int_0^{\mathrm{t}} \kappa(t-\tau)\frac{\mathrm{d}E(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{14}$$

To calculate the time-dependent Poisson's ratio $\nu_{\mathrm{r}}(t)$ using (14), one needs bulk and tensile data over the full range of time of interest to evaluate the above convolution integral. In applications, expressions for stress or deformation fields contain convolution integrals involving a physical quantity and Poisson's ratio. In view of (14), nested convolutions arise. Their evaluation is often done numerically and can lead to time-consuming computations. As will be shown in Section 2.3 below, if stress or deformation fields can be expressed in terms of the dynamic frequency dependent Poisson's ratio, then evaluation of the expressions requires only algebraic operations.

2.2. Creep in Tension

Consider creep in uniaxial tension: A stress of magnitude σ_{o} is applied as a step function of time: $\sigma_{\mathrm{L}}(t) = \sigma_0 H(t)$. The longitudinal strain becomes time dependent in a viscoelastic material:

$$\varepsilon_{\mathrm{L}}(t) = \sigma_{\mathrm{o}} J_E(t), \tag{15}$$

with $J_E(t)$ as the creep compliance for axial deformation. Substitute the Laplace transform of the relation in (15) into (5). This gives the relation,

$$\underline{\varepsilon}_{\mathrm{tr}}(p) = -\frac{1}{2}\underline{\varepsilon}_{\mathrm{L}}(p) + \frac{1}{6}\kappa\sigma_{\mathrm{o}} p \underline{E}(p) \underline{J}_E(p). \tag{16}$$

It is a standard result in viscoelasticity theory that the interrelation following a Laplace transformation is

$$p^2 \underline{E}(p) \underline{J}_E(p) = 1. \tag{17}$$

Using this result, and following the correspondence principle, applying an inverse Laplace transform gives the following relation in the time domain,

$$\varepsilon_{\mathrm{tr}}(t) = -\frac{1}{2}\varepsilon_{\mathrm{L}}(t) + \frac{1}{6}\kappa\sigma_{\mathrm{o}} H(t). \tag{18}$$

Dividing the above transverse strain by the longitudinal strain and recognizing the ratio as $\nu_{\mathrm{c}}(t)$, the Poisson's ratio in creep, the following is obtained:

$$\nu_{\mathrm{c}}(t) = -\frac{\varepsilon_{\mathrm{tr}}(t)}{\varepsilon_{\mathrm{L}}(t)} = \frac{1}{2} - \frac{1}{6}\kappa\sigma_{\mathrm{o}}\frac{H(t)}{\varepsilon_{\mathrm{L}}(t)}. \tag{19}$$

Recall that the creep compliance is $J_E(t) = \varepsilon_{\mathrm{L}}(t)/\sigma_{\mathrm{o}}$. Using the relation in (15) and recognizing that $t > 0$, the Poisson's ratio in creep is

$$\nu_{\mathrm{c}}(t) = -\frac{\varepsilon_{\mathrm{tr}}(t)}{\varepsilon_{\mathrm{L}}(t)} = \frac{1}{2} - \frac{1}{6}\kappa\frac{1}{J_E(t)}. \tag{20}$$

Observe that the Poisson's ratio $\nu_{\mathrm{c}}(t)$ in creep differs from the Poisson's ratio $\nu_{\mathrm{r}}(t)$ in relaxation given in (7), since $E(t) \neq 1/J_E(t)$. Specifically [1], $E(t)J_E(t) \leq 1$ with the equality corresponding to an elastic solid. From (7) and (20), the interrelation between $\nu_{\mathrm{c}}(t)$ and $\nu_{\mathrm{r}}(t)$ is written as follows in terms of $E(t)J_E(t)$:

$$\nu_{\mathrm{c}}(t) = \nu_{\mathrm{r}}(t) + \frac{1}{6}\kappa E(t)\left[1 - \frac{1}{E(t)J_E(t)}\right]. \tag{21}$$

Observe also that with a constant bulk modulus assumed in (7) and (20), $\nu_{\mathrm{r}}(t)$ and $\nu_{\mathrm{c}}(t)$ are both increasing functions of time. As will be shown below, the Poisson's ratio need not increase for all materials; several counter-examples are given. Moreover, since [1, 3, 4] $E(t)J_E(t) \leq 1$, (21) implies $\nu_c(t) < \nu_r(t)$.

To make a numerical comparison of Poisson's ratios in relaxation and creep, consider $E(t)$ proportional to a power law in time, t^{-n}. This is a convenient form since in linear viscoelasticity, it can be used to obtain simple exact analytical interrelations for relaxation, creep and mechanical damping tan δ. The value of n is a dimensionless measure of the magnitude of viscoelastic effects since it represents the slope of creep or relaxation curves (on a doubly logarithmic plot) and it is proportional to the mechanical damping. For example, the interrelation between relaxation and creep is $E(t)J_E(t) = \sin n\pi/n\pi$, and this form can be substituted in (21). Suppose, for example, $n = 0.1$. For power law relaxation, the usual linear interrelation [3, 4] gives $\tan\delta = \tan n\pi/2$; for $n = 1$ this is $\tan\delta = 0.16$. The quantity $[n\pi/\sin n\pi - 1]$ is 0.0166, so if $\nu_r(t) = 0.3$ at a given time, corresponding (from (3)) to $\kappa E = 3(1 - 2\nu) = 1.2$, then, via (21) $\nu_c(t)$ differs from $\nu_r(t)$ by 1%. The second term in (7) and (20) is half the total, therefore the difference between the time dependent part of Poisson's ratio in creep and relaxation is 2%. The example $n = 0.1$ represents a higher degree of viscoelasticity than is ordinarily seen in the glassy regime of polymers or in soft metals yet the Poisson's ratios in relaxation and in creep do not differ by much. The difference between the time dependent Poisson's ratio in relaxation and creep increases with the degree of viscoelasticity as quantified by the slope n in the relaxation curve in a logarithmic plot.

The following comparison between Poisson's ratios in creep and relaxation makes use of a three parameter solid. This idealized solid is modeled by two springs and a viscous damper; the solid exhibits exponential creep or relaxation and in the frequency domain, it exhibits a peak in the damping. As with the power law model used above, analytical forms for creep, relaxation and damping are well known. The relaxation function $E(t)$ is

$$E(t) = E_2 + E_1 e^{-t/\tau_r} = E_2\left[1 + \Delta e^{-t/\tau_r}\right], \tag{22}$$

in which the *relaxation strength* Δ is defined as the change in stiffness during relaxation divided by the stiffness at long time,

$$\Delta = \frac{E_1}{E_2} = \frac{E(0) - E(\infty)}{E(\infty)}. \tag{23}$$

We remark that if Δ is small, the peak value of the mechanical damping tan δ is $\frac{1}{2}\Delta$. The ratio of retardation (creep) time τ_c to relaxation time τ_r is $\tau_c = \tau_r(1 + \Delta)$ and it depends on the *relaxation strength* Δ.

Using (17), the corresponding creep function is

$$J_E(t) = \frac{1}{E_2}\left[1 - \frac{\Delta}{1 + \Delta} e^{-t/\{\tau_r(1+\Delta)\}}\right]. \tag{24}$$

Results for $\nu_r(t)$ and $\nu_c(t)$ via (7) and (20), respectively, are plotted in Figure 1 assuming $\tau_r = 1$ s. For Figure 1a, $\Delta = 0.1$ and $E_2 = 1.15/\kappa$, corresponding to a β peak in the damping of a glassy polymer or to an order–disorder or twin boundary peak in a metal. Here the curves for Poisson's ratio in creep and relaxation are indistinguishable. For Figure 1b, $\Delta = 1$ and $E_2 = 0.6/\kappa$. Even with such a large relaxation strength the Poisson's ratio curves do not differ by much. For Figure 1c, $\Delta = 1{,}000$ and $E_2 = 0.001/\kappa$. This corresponds to the glass–rubber transition in a

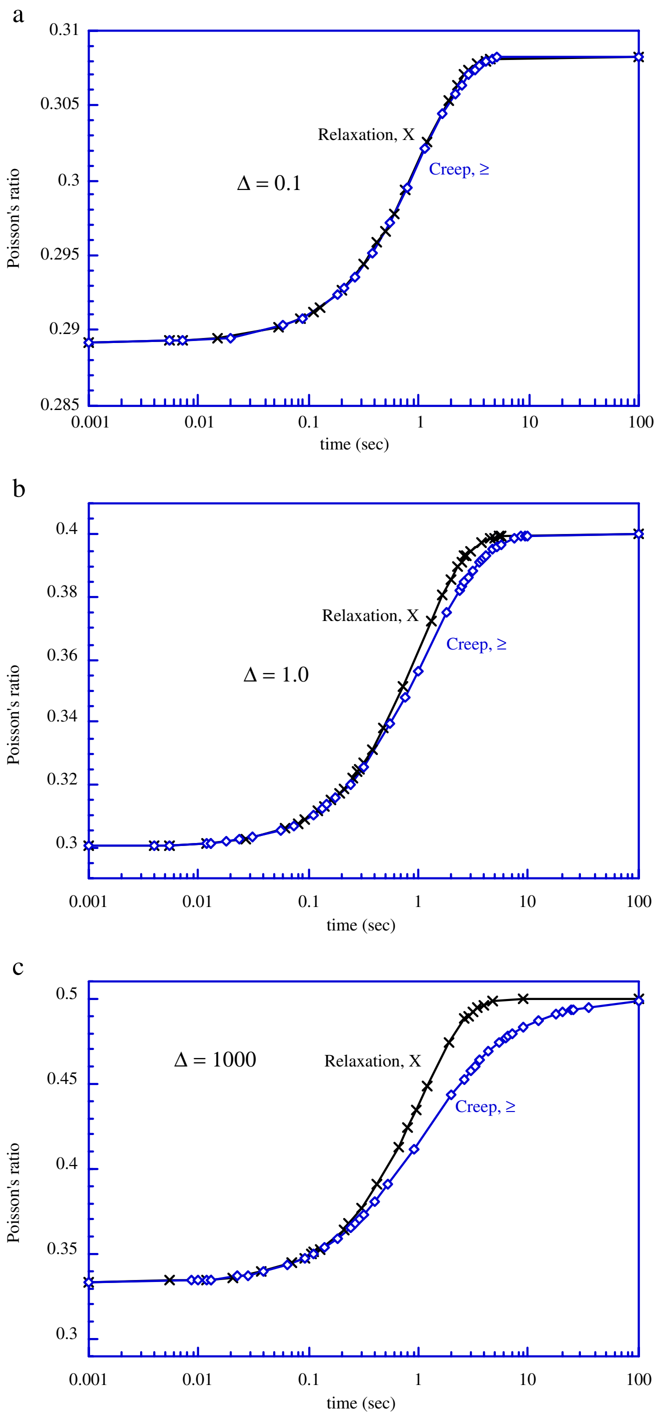

Figure 1 Poisson's ratio in relaxation and in creep for a single exponential relaxation with time constant 1 s. **a** Relaxation strength $\Delta = 0.1$. **b** Relaxation strength $\Delta = 1.0$. **c** Relaxation strength $\Delta = 1{,}000$.

polymer. As one approaches the rubbery (large time) regime, the Poisson's ratio for creep changes more slowly than the Poisson's ratio for relaxation. The creep Poisson's ratio would present some experimental challenges in this case since the longitudinal creep strain changes by a factor of 1,000 through the transition. This change in strain would tax the available dynamic range in an experiment, since considerable precision is required to resolve the small difference between creep and relaxation Poisson's ratio. Thus, from various points of view, the difference between Poisson's ratios in creep and relaxation is minor.

2.3. Dynamic Behavior in Tension

Suppose the longitudinal strain is a sinusoidal function of time, $\varepsilon_{\mathrm{L}} = \varepsilon_{\mathrm{L}0} \sin \omega t$ with ω as the angular frequency in radians per second. Material properties of viscoelastic materials under dynamic loading are expressed as complex quantities in which the imaginary part is associated with a phase shift and with energy dissipation. If a solution to an elasticity problem is known, the solution to the corresponding problem for a viscoelastic material can be obtained via the dynamic correspondence principle [10]. In this approach, each elastic constant is replaced with the corresponding complex quantity. No inverse transform is used here since the complex moduli have a direct physical interpretation. Application of the dynamic correspondence principle to (6) gives the following, with E^* as the complex dynamic Young's modulus.

$$\varepsilon_{\mathrm{tr}} = -\left[\frac{1}{2} - \frac{1}{6}\kappa E^*\right]\varepsilon_{\mathrm{L}}. \tag{25}$$

The ratio of the imaginary part of E^* to the real part is tan δ with δ as the phase difference between stress and strain. Define the complex dynamic Poisson's ratio ν^* as the ratio of transverse to longitudinal strain,

$$\nu^* = -\frac{\varepsilon_{\mathrm{tr}}}{\varepsilon_{\mathrm{L}}}, \tag{26}$$

in which these strains may have a phase difference. Then

$$\nu^* = \left[\frac{1}{2} - \frac{1}{6}\kappa E^*\right]. \tag{27}$$

The phase angle δ depends only on the material; it is independent of whether one controls the displacement or the load.

The dynamic frequency domain approach has the further advantage of simplicity [11] in that viscoelasticity in the bulk compliance can be readily incorporated by replacing (using the dynamic correspondence principle) κ with κ^*. Equation (27) becomes

$$\nu^* = \left[\frac{1}{2} - \frac{1}{6}\kappa^* E^*\right]. \tag{28}$$

This is an algebraic relation in the bulk and tensile properties. By contrast, in the time domain approach, a nested convolution integral is required as presented above in (13). Indeed, in the dynamic formulation it is evident that a real Poisson's ratio in the frequency domain (corresponding to a time-independent Poisson's ratio in the time domain) can occur only if there is an exact balance between the phase angle in the axial properties and the phase angle in the bulk properties. In the case of polymers, which exhibit little viscoelasticity in bulk properties in comparison to shear, a real dynamic Poisson's ratio (one constant in the time domain) is not to be expected, in agreement with the analysis of Hilton [5].

2.4. Bending

Consider the role of Poisson's ratio ν in the field of displacements (u_x,u_y,u_z) for pure bending of an isotropic elastic beam [12]. The bending moment is about the y-axis, the bending stresses vary along the x- and z-axes along the beam. The radius of curvature is denoted by R. The displacement field is given by,

$$\left\{u_{\mathrm{x}} = -(1/2R)\left[z^2 + \nu(x^2 - y^2)\right], u_y = -\nu xy/R, u_z = xz/R\right\}. \tag{29}$$

As in (3) write the Poisson's ratio in terms of the moduli to facilitate use of the correspondence principle. Then (29) becomes,

$$\left\{u_x = -(1/2R)\left[z^2 + \left(\frac{1}{2} - \frac{1}{6}\kappa E\right)(x^2 - y^2)\right], u_y = -\left(\frac{1}{2} - \frac{1}{6}\kappa E\right)xy/R, u_z = xz/R\right\}. \tag{30}$$

Apply the correspondence principle to the displacement u_y; $\Omega = 1/R$ is the curvature. As in Sections 2.1 and 2.2, bulk relaxation is neglected. The following is obtained:

$$\underline{u}_y = -\underline{\Omega}\left(\frac{1}{2} - \frac{1}{6}\kappa p \underline{E}(p)\right)xy. \tag{31}$$

Transforming to the time domain, this becomes:

$$u_y(t) = \left[-\frac{1}{2}\Omega(t) + \frac{1}{6}\kappa \int_0^{\mathrm{t}} E(t-\tau)\frac{\mathrm{d}\Omega(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau\right]xy. \tag{32}$$

In a bending relaxation experiment, the curvature is a step function $\Omega(t) = \Omega_0 H(t)$, so

$$u_y(t) = \Omega_0\left[-\frac{1}{2} + \frac{1}{6}\kappa E(t)\right]xy. \tag{33}$$

This may be written in terms of a bending relaxation Poisson's ratio $\nu_{\mathrm{b}}(t)$,

$$u_y(t) = -\Omega_0[\nu_{\mathrm{b}}(t)]xy. \tag{34}$$

Observe that this Poisson's ratio is identical to the one given in (7) for axial relaxation so $\nu_{\mathrm{b}}(t) = \nu_{\mathrm{r}}(t)$. Therefore the tilt $\mathrm{d}u_y/\mathrm{d}x$ of the lateral surfaces in a bending test can be directly interpreted in the context of a viscoelastic Poisson's ratio. The tilt is readily measurable by determining the angular displacement of a laser beam reflected off either a polished specimen surface or a mirror attached to the specimen. The angular motion of the reflected light beam can be readily converted to an electrical signal by illuminating a split diode detector with the light.

It is evident from (29) that the anticlastic curvature in the first term can be handled in the same way. Specifically for a bar of depth 2a in the direction of bending, and width 2b, the displacement at the top surface of the cross-section in the direction normal to the surface is:

$$u_x\,|_{x=a} = -(1/2R)\left[z^2 + (a^2 - y^2)\nu_{\mathrm{r}}(t)\right]. \tag{35}$$

The contours of constant displacement are hyperbolic and time dependent. The angle α between an asymptote to the hyperbolae and the z-axis along the beam is given by $\tan\,\alpha(t) = 1\sqrt{\nu_{\mathrm{r}}}(t)$. Measurement of the variation of this angle with time can be used to determine $\nu_{\mathrm{r}}(t)$. For example, Timoshenko and Goodier [12] describe a method of determining Poisson's ratio via classical interferometry to observe these contours in the displacement in a polished specimen. One could use also holographic interferometry to do a similar measurement in a specimen with a rough surface, as was done with negative Poisson's ratio foam [13].

3. The Correspondence Principle for Poisson's Ratio

3.1. Transient Properties

The correspondence principle for the time dependent Poisson's ratio $\nu(t)$ is derived as follows. The rationale is to develop a method which is applicable to Poisson's ratio as the standard correspondence principle is applicable to the moduli. The constitutive relation for the elastic case is $\varepsilon_{11} = \frac{1}{E}\{\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\}$. This elementary form is developed using superposition. Superposition is used in the following to develop the correspondence principle for Poisson's ratio in linear viscoelasticity.

Suppose σ_{11} is not a step stress, but varies arbitrarily with time. Decompose the stress into the superposition of a sequence of step stresses each applied at a different time. The corresponding longitudinal ε_{11} strain at time t is the superposition or addition of the separate responses due to all the step stresses applied up to time t. If $\mathrm{d}\sigma_{11}(\tau)$ is a step stress applied at time τ, its corresponding longitudinal strain at time t is $J_E(t-\tau)\mathrm{d}\sigma_{11}(\tau)$. The longitudinal strain at time t due to all the steps is

$$\varepsilon_{11}(t) = \int_0^{\mathrm{t}} J_E(t-\tau)\mathrm{d}\sigma_{11}(\tau). \tag{36}$$

This has the form of a convolution. Gurtin and Sternberg [14] use the notation $\varepsilon_{11} = J_E \times \mathrm{d}\sigma_{11}$ to represent (36). Note that the Laplace transform of (36) is

$$\underline{\varepsilon}_{11}(p) = p\underline{J}_E(p)\underline{\sigma}_{11}(p). \tag{37}$$

Since the Laplace transforms of the creep compliance and the relaxation function are related by

$$p^2\underline{E}(p)\underline{J}_E(p) = 1, \tag{38}$$

Equation (37) can be written

$$\underline{\varepsilon}_{11}(p) = \underline{\sigma}_{11}(p)\big/p\underline{E}(p). \tag{39}$$

This is analogous, with E replaced by $p\underline{\mathrm{E}}(p)$, to the corresponding relation in linear elasticity

$$\varepsilon_{11} = \sigma_{11}\big/E. \tag{40}$$

Stress σ_{11} also produces transverse strains ε_{22} and ε_{33}. Let $\varepsilon_{\mathrm{tr}}$ denote either of these transverse strains and let $\varepsilon_{\mathrm{L}} = \varepsilon_{11}$ denote the longitudinal strain. The expression for transverse strain $\varepsilon_{\mathrm{tr}}$ is constructed using the same superposition ideas used to get (36).

First, consider the transverse strain due to a step longitudinal strain ${\varepsilon_{\mathrm{L}}}^0$. Let $\nu_{\mathrm{r}}(t)$ be the Poisson ratio defined by

$$\nu_{\mathrm{r}}(t) = -\frac{\varepsilon_{\mathrm{tr}}(t)}{\varepsilon_{\mathrm{L}}^0}, \tag{41}$$

that is, the response in a stress relaxation test. If ε_{L} is not a step longitudinal strain, but varies with time, it is decomposed into the superposition of a sequence of step longitudinal strains, each applied at a different time. The corresponding transverse strain $\varepsilon_{\mathrm{tr}}(t)$ at time t is the superposition or addition of the separate responses due to the all the step longitudinal strains applied up to time t. If $\mathrm{d}\varepsilon_{\mathrm{L}}(\tau)$ is a step longitudinal strain applied at time τ, its corresponding transverse strain at time t is $-\nu_{\mathrm{r}}(t-\tau)\mathrm{d}\varepsilon_{\mathrm{L}}(\tau)$. The transverse strain at time t due to all the steps is

$$\varepsilon_{\mathrm{tr}}(t) = -\int_0^{\mathrm{t}} \nu_{\mathrm{r}}(t-\tau)\mathrm{d}\varepsilon_{\mathrm{L}}(\tau). \tag{42}$$

As above, this may be written in the Gurtin–Sternberg notation as $\varepsilon_{\mathrm{tr}} = -\nu_{\mathrm{r}} \times \mathrm{d}\varepsilon_{\mathrm{L}}$. This can be expressed in terms of the stress by using nested convolutions, $\varepsilon_{\mathrm{tr}} = -\nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma_{11}$. Note that the Laplace transform of (42) is

$$\underline{\varepsilon}_{\mathrm{tr}}(p) = -p\underline{\nu}_{\mathrm{r}}(p)\underline{\varepsilon}_{\mathrm{L}}(p). \tag{43}$$

This can be rewritten

$$\underline{\varepsilon}_{\mathrm{tr}}(p) = -\frac{p\underline{\nu}_{\mathrm{r}}(p)}{p\underline{E}(p)}\underline{\sigma}_{11}(p), \tag{44}$$

which is analogous, with E replaced by $p\underline{\mathrm{E}}(p)$, and with ν replaced by $p\underline{\nu}_r(p)$, to the corresponding relation in linear elasticity:

$$\varepsilon_{\mathrm{tr}} = -\frac{\nu}{E}\sigma_{11}. \tag{45}$$

Via superposition, the total ε_{11} strain is written as a sum of contributions due to all of the normal stress components. The first term is a longitudinal strain given by (36). The last two terms are transverse strains given by combining (42) and (36).

$$\varepsilon_{11} = \varepsilon_{11}(\sigma_{11}) + \varepsilon_{11}(\sigma_{22}) + \varepsilon_{11}(\sigma_{33}) \tag{46}$$

Using the convolution notation, (46) can be written as

$$\varepsilon_{11} = J_E \times \mathrm{d}\sigma_{11} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma_{22} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma_{33} \tag{47}$$

In a similar manner,

$$\varepsilon_{22} = \mathrm{J}_E \times \mathrm{d}\sigma_{22} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma_{11} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma_{33}, \tag{48}$$

$$\varepsilon_{33} = J_E \times \mathrm{d}\sigma_{33} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma_{11} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma_{22}. \tag{49}$$

Taking the Laplace transform of (47) and using (39) and (44) we get

$$\underline{\varepsilon}_{11}(p) = \frac{\underline{\sigma}_{11}(p)}{p\underline{E}(p)} - \frac{p\underline{\nu}_{\mathrm{r}}(p)}{p\underline{E}(p)}\sigma_{22}(p) - \frac{p\underline{\nu}_{\mathrm{r}}(p)}{p\underline{E}(p)}\sigma_{33}(p), \tag{50}$$

or

$$\underline{\varepsilon}_{11}(p)=\frac{1}{p\underline{E}(p)}\left\{\underline{\sigma}_{11}(p)-p\underline{\nu}_{\mathrm{r}}(p)\underline{\sigma}_{22}(p)-p\underline{\nu}_{\mathrm{r}}(p)\underline{\sigma}_{33}(p)\right\}. \tag{51}$$

In a similar manner, the strains in the other directions are:

$$\underline{\varepsilon}_{22}(\mathrm{p})=\frac{1}{\mathrm{p}\underline{E}(\mathrm{p})}\left\{\underline{\sigma}_{22}(\mathrm{p})-\mathrm{p}\underline{\nu}_{\mathrm{r}}(\mathrm{p})\underline{\sigma}_{11}(\mathrm{p})-\mathrm{p}\underline{\nu}_{\mathrm{r}}(\mathrm{p})\underline{\sigma}_{33}(\mathrm{p})\right\}, \tag{52}$$

$$\underline{\varepsilon}_{33}(p)=\frac{1}{p\underline{E}(p)}\left\{\underline{\sigma}_{33}(p)-p\underline{\nu}_{\mathrm{r}}(p)\underline{\sigma}_{11}(p)-p\underline{\nu}_{\mathrm{r}}(p)\underline{\sigma}_{22}(p)\right\}. \tag{53}$$

Similar remarks apply to the relation between shear strain $\gamma_{12}(t)=2\varepsilon_{12}(t)$ and shear stress $\sigma_{12}(t)$ and the creep compliance $J_G(t)$ in shear:

$$\gamma_{12}(t)=\int_0^{\mathrm{t}} J_G(t-\tau)\frac{\mathrm{d}\sigma_{12}}{\mathrm{d}\tau}\mathrm{d}\tau, \tag{54}$$

for which the Laplace transform is

$$\underline{\gamma}_{12}(p)=p\underline{J}_G(p)\underline{\sigma}_{12}(p). \tag{55}$$

Equations (51)–(53) have the same form as the three-dimensional constitutive equation for isotropic linear elasticity with E replaced by $p\underline{E}(p)$, and with ν replaced by $p\underline{\nu}_r(p)$. This is the correspondence principle for Poisson's ratio in transient form. This development shows that the Poisson's ratio to be used is the relaxation form.

3.2. Dynamic Properties

The dynamic correspondence principle for Poisson's ratio is developed in the following. Consider in this section a particular stress history in which the stresses vary sinusoidally with time. Suppose the only stress is $\sigma_{11}(t)=\sigma_{11}^{\mathrm{o}}e^{\mathrm{i}\omega t}$. Substitute this into (36) which is valid for any stress history. After sufficient time, transients die out and the strain is given by

$$\varepsilon_{11}(t)=\varepsilon_{11}^{\mathrm{o}}e^{\mathrm{i}\omega t}=J_E{}^*(\omega)\sigma_{11}^{\mathrm{o}}e^{\mathrm{i}\omega t}. \tag{56}$$

in which $J_E{}^*(\omega)$ is the complex dynamic compliance in extension, a function of the angular frequency ω. Dynamic properties can be measured directly in experiments or calculated via Fourier transformation from transient properties. The amplitudes of stress and strain are related by

$$\varepsilon_{11}^{\mathrm{o}}=J_E{}^*(\omega)\sigma_{11}^{\mathrm{o}}. \tag{57}$$

The dynamic Young's modulus $E^*(\omega)$ is related to the dynamic compliance by $J_E{}^*(\omega)=1/E^*(\omega)$. Equation (57) thus becomes

$$\varepsilon_{11}^{\mathrm{o}}=\left[1/E^*(\omega)\right]\sigma_{11}^{\mathrm{o}}. \tag{58}$$

This is analogous to the corresponding relation in linear elasticity $\varepsilon_{11}=\sigma_{11}/E$.

Substitute the sinusoidally varying strain in (56) into (42). After the transients die out, the transverse strain becomes sinusoidal and is given in terms of a complex Poisson's ratio $\nu^*(\omega)$ by

$$\varepsilon_{\mathrm{tr}}(t)=\varepsilon_{\mathrm{tr}}^{\mathrm{o}}e^{\mathrm{i}\omega t}=-\nu^*(\omega)\varepsilon_{\mathrm{L}}^{\mathrm{o}}e^{\mathrm{i}\omega t}=-\nu^*(\omega)\varepsilon_{11}^{\mathrm{o}}e^{\mathrm{i}\omega t} \tag{59}$$

The complex Poisson's ratio incorporates the possibility that the longitudinal and transverse strains can have a phase shift. The amplitudes of the transverse and longitudinal strains are related by

$$\varepsilon_{\mathrm{tr}}^{\mathrm{o}} = -\nu^*(\omega)\varepsilon_{11}^{\mathrm{o}}. \tag{60}$$

Combining (58) and (60) gives

$$\varepsilon_{\mathrm{tr}}^{\mathrm{o}} = -\frac{\nu^*(\omega)}{E^*(\omega)}\sigma_{11}^{\mathrm{o}}. \tag{61}$$

This is analogous to the corresponding relation in linear elasticity, (45).

Now, consider the case in which all three normal stresses vary sinusoidally, $\sigma_{11}(t) = \sigma_{11}^{\mathrm{o}} e^{\mathrm{i}\omega t}$, $\sigma_{22}(t) = \sigma_{22}^{\mathrm{o}} \mathrm{e}^{\mathrm{i}\omega t}$, $\sigma_{33}(t) = \sigma_{33}^{\mathrm{o}} e^{\mathrm{i}\omega t}$. The total strain $\varepsilon_{11}(t)$ is determined by superposition, just as in the transient case discussed in Sections 3.1:

$$\varepsilon_{11}^{\mathrm{o}} = \frac{1}{E^*(\omega)}\left[\sigma_{11}^{\mathrm{o}} - \nu^*(\omega)\sigma_{22}^{\mathrm{o}} - \nu^*(\omega)\sigma_{33}^{\mathrm{o}}\right]. \tag{62}$$

Similarly,

$$\varepsilon_{22}^{\mathrm{o}} = \frac{1}{E^*(\omega)}\left[\sigma_{22}^{\mathrm{o}} - \nu^*(\omega)\sigma_{11}^{\mathrm{o}} - \nu^*(\omega)\sigma_{33}^{\mathrm{o}}\right], \tag{63}$$

$$\varepsilon_{33}^{\mathrm{o}} = \frac{1}{E^*(\omega)}\left[\sigma_{33}^{\mathrm{o}} - \nu^*(\omega)\sigma_{11}^{\mathrm{o}} - \nu^*(\omega)\sigma_{22}^{\mathrm{o}}\right]. \tag{64}$$

These have the same form as the three-dimensional constitutive equations for linear isotropic elasticity, and they form the basis for the dynamic correspondence principle for Poisson's ratio.

4. Consequences of Superposition in Isotropic Solids

4.1. Relation between Young's and Shear Moduli

In this section, the notion of superposition is used to formally, by direct construction, extend several results in the theory of isotropic elasticity to the viscoelastic domain. First, the meaning of the relation $E = 2\,G\,(1 + \nu)$ for a viscoelastic solid is developed using superposition. The result is then compared with the result obtained via the correspondence principle developed above.

Let $x_1{}', x_2{}', x_3{}'$ coordinate axes be obtained by rotating the x_1, x_2, x_3 axes about the x_3 axis by an angle θ. The strain with respect to the rotated primed axes is denoted $\varepsilon_{ij}{}'$ and the stress is denoted $\sigma_{ij}{}'$. If the material is isotropic, stress and strain are related by the same equations in the rotated system as in the original system. By (47)–(49) and (54), the following are obtained:

$$\varepsilon'_{11} = J_E \times \mathrm{d}\sigma'_{11} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma'_{22} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma'_{33}, \tag{65}$$

$$\varepsilon'_{22} = J_E \times \mathrm{d}\sigma'_{22} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma'_{11} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma'_{33}, \tag{66}$$

$$\varepsilon'_{33} = J_E \times \mathrm{d}\sigma'_{33} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma'_{11} - \nu_{\mathrm{r}} \times J_E \times \mathrm{d}\sigma'_{22}, \tag{67}$$

$$\gamma'_{12} = J_G \times \mathrm{d}\sigma'_{12}. \tag{68}$$

Consider a state of plane stress where $\sigma'_{33} = \sigma_{33} = 0$ and $\sigma'_{32} = \sigma_{32} = 0$. The tensor transformation laws for stress and strain hold at each value of time. Thus,

$$\sigma'_{11}(t) = \sigma_{11}(t)\cos^2\theta + 2\sigma_{12}(t)\sin\theta\cos\theta + \sigma_{22}(t)\sin^2\theta, \tag{69}$$

$$\sigma'_{22}(t) = \sigma_{11}(t)\sin^2\theta + 2\sigma_{12}(t)\sin\theta\cos\theta + \sigma_{22}(t)\cos^2\theta, \tag{70}$$

$$\sigma'_{12}(t) = (\sigma_{22}(t) - \sigma_{11}(t))\sin\theta\cos\theta + \sigma_{12}(t)\big(\cos^2\theta - \sin^2\theta\big), \tag{71}$$

and

$$\varepsilon'_{11}(t) = \varepsilon_{11}(t)\cos^2\theta + \gamma_{12}(t)\sin\theta\cos\theta + \varepsilon_{22}(t)\sin^2\theta, \tag{72}$$

$$\varepsilon'_{22}(t) = \varepsilon_{11}(t)\sin^2\theta - \gamma_{12}(t)\sin\theta\cos\theta + \varepsilon_{22}(t)\cos^2\theta, \tag{73}$$

$$\gamma'_{12}(t) = 2(\varepsilon_{22}(t) - \varepsilon_{11}(t))\sin\theta\cos\theta + \gamma_{12}(t)\big(\cos^2\theta - \sin^2\theta\big). \tag{74}$$

Let $\sigma'_{33} = \sigma_{33} = 0$ in (65) and (47) and (48). Substitute (72) into the left side of (65), (69) and (70) into the right side of (65) to obtain

$$\begin{aligned}&\varepsilon_{11}(t)\cos^2\theta + \gamma_{12}(t)\sin\theta\cos\theta + \varepsilon_{22}(t)\sin^2\theta\\ &\quad= J_E \times \mathrm{d}\big(\sigma_{11}(t)\cos^2\theta + 2\sigma_{12}(t)\sin\theta\cos\theta + \sigma_{22}\sin^2\theta\big)\\ &\qquad -\nu_{\mathrm{r}} \times \mathrm{d}\big(J_E \times \mathrm{d}\big(\sigma_{11}(t)\sin^2\theta - 2\sigma_{12}(t)\sin\theta\cos\theta + \sigma_{22}(t)\cos^2\theta\big)\big).\end{aligned} \tag{75}$$

Since θ is independent of time, this can be rewritten

$$\begin{aligned}&\varepsilon_{11}(t)\cos^2\theta + \gamma_{12}(t)\sin\theta\cos\theta + \varepsilon_{22}(t)\sin^2\theta\\ &\quad= \cos^2\theta\big[J_E \times \mathrm{d}\sigma_{11} - \nu_{\mathrm{r}} \times \mathrm{d}(J_E \times \mathrm{d}\sigma_{22})\big] + \sin^2\theta\big[J_E \times \mathrm{d}\sigma_{22} - \nu_{\mathrm{r}} \times \mathrm{d}(J_E \times \mathrm{d}\sigma_{11})\big]\\ &\qquad + \sin\theta\cos\theta\big[J_E \times \mathrm{d}(2\sigma_{12}) + \nu_{\mathrm{r}} \times \mathrm{d}(J_E \times \mathrm{d}(\sigma_{12}))\big].\end{aligned} \tag{76}$$

By (47) and (48) with $\sigma_{33} = 0$, the terms in $\sin^2\theta$ and $\cos^2\theta$ cancel out, so (76) reduces to

$$\gamma_{12}(t)\sin\theta\cos\theta = \sin\theta\cos\theta\big[J_E \times \mathrm{d}(2\sigma_{12}) + \nu_{\mathrm{r}} \times \mathrm{d}(J_E \times \mathrm{d}(\sigma_{12}))\big]. \tag{77}$$

Since $\sin\theta\cos\theta$ cancels then this must hold for any angle, i.e.,

$$\gamma_{12}(t) = J_E \times \mathrm{d}(2\sigma_{12}) + \nu_{\mathrm{r}} \times \mathrm{d}(J_E \times \mathrm{d}(\sigma_{12})). \tag{78}$$

Using (54) b, we find

$$J_G \times \mathrm{d}\sigma_{12} = J_E \times \mathrm{d}(2\sigma_{12}) + \nu_{\mathrm{r}} \times \mathrm{d}(J_E \times \mathrm{d}(\sigma_{12})). \tag{79}$$

Take the Laplace transform to obtain

$$p\underline{J}_G(p)\underline{\sigma}_{12}(p) = 2p\underline{J}_E(p)\underline{\sigma}_{12}(p) + 2p\underline{\nu}_{\mathrm{r}}(p)\big(p\underline{J}_E(p)\underline{\sigma}_{12}(p)\big). \tag{80}$$

This must hold for all stress histories, so

$$\underline{J}_G(p) = 2\big[1 + p\underline{\nu}_{\mathrm{r}}(p)\big]\underline{J}_E(p). \tag{81}$$

A relation between the transforms of the creep compliances in shear and extension and Poisson's ratio is next developed. Using (38), (81) can be converted from the creep to the relaxation properties,

$$\underline{E}(p) = 2\big[1 + p\underline{\nu}_{\mathrm{r}}(p)\big]\underline{G}(p). \tag{82}$$

The inverse transformation of (82) gives the relation in the time domain,

$$E(t) = 2G(t) + 2\int_{0}^{t} \nu_{\mathrm{r}}(t-\tau)\frac{\mathrm{d}G}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{83}$$

This form, obtained by direct construction via superposition, is identical to that obtained via the correspondence principle from the elastic relation $E = 2\,G\,(1+\nu)$, provided the time dependent Poisson's ratio is interpreted as the relaxation form $\nu_{\mathrm{r}}(t)$.

4.2. The Constrained Modulus C_{1111}

For elastic solids the constrained modulus C_{1111} is given by [8]

$$C_{1111} = E\frac{1-\nu}{(1+\nu)(1-2\nu)} = 2G\frac{1-\nu}{(1-2\nu)}. \tag{84}$$

G and C_{1111} can be measured upon the same specimen with ultrasound and C_{1111} can be measured via constrained compression. Poisson's ratio is not usually measured in this context but it could be done by measuring the wall constraint force in constrained compression. One could infer the Poisson's ratio from G and C_{1111}; in the context of wave propagation, one obtains a complex Poisson's ratio ν^* via the dynamic correspondence principle. In the case of polymers C_{1111} changes much less with time or frequency through the glass–rubber transition than G; as G decreases with time, the Poisson's ratio term on the right correspondingly increases. This is an example of a large effect of time/frequency dependence of Poisson's ratio; neglect of such effects would lead to an erroneous conclusion that longitudinal waves are as dispersive (frequency dependent in their velocity) as shear waves.

5. Direction of Change of Poisson's Ratio

5.1. Monotonicity

Several systems are known which exhibit a Poisson's ratio which increases with time. For example, in the glass to rubber transition of polymers the shear modulus may change by three orders of magnitude but the bulk modulus changes by about a factor of two [3]. This corresponds in isotropic solids to an increase in Poisson's ratio from about 0.3 to nearly 0.5 with time. Similarly, there is a difference between shear deformation and deformation containing a substantial volumetric component in geological materials, in which shear waves undergo much more attenuation than compressional waves [15] and therefore more creep or relaxation by virtue of the interrelations among viscoelastic properties. Tschoegl et al. [6], possibly in view of the above observations,suggest that such a monotonic increase in Poisson's ratio is required by the theory of viscoelasticity. We show here that Poisson's ratio can increase or decrease with time, or be constant in time, depending on the material system; moreover the Poisson's ratio need not be monotonic in time.

A Poisson's ratio that is constant in time occurs in low-density honeycombs and foams [16]. Poisson's ratio is governed by the cell geometry, specifically the angle between cell ribs. For small deformation, this angle is essentially constant, so

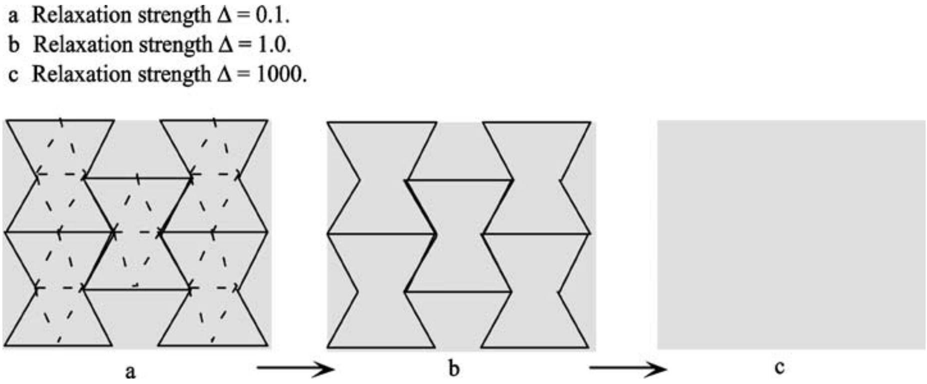

Figure 2 Honeycomb lattice with time varying Poisson's ratio. The *gray shading* is compliant foam filling. Ribs drawn as *dash lines* relax at a short time $\tau_{\text{dash}} = 0.01$ s. Then at a much longer time, the ribs drawn as *solid lines* relax $\tau_{\text{solid}} = 10$ s.

Poisson's ratio is constant in time. It is assumed the deformation is sufficiently slow that viscous resistance of air in the pores is negligible. One may also interpret this in the context of $E(t)$ decreasing with time and the bulk compliance $\kappa(t)$ increasing with time so Poisson's ratio is constant in time.

A Poisson's ratio which decreases with time occurs in a designed lattice structure as follows. Consider a two-dimensional triangular lattice system which has a positive Poisson's ratio of 0.25 (for equilateral triangles) [17]. If the stiffness of selected rib elements in the lattice is allowed to relax to zero, the remaining ligaments form (at long times) a re-entrant lattice [18, 19] with a negative Poisson's ratio. A generalization of this notion is shown in Figure 2. The honeycomb is assumed to be filled with a compliant foam (shown as gray in the diagram) of modulus much less than that of the honeycomb ribs. At sufficiently short time t, all the honeycomb ribs have the same modulus so the resulting triangular honeycomb has a Poisson's ratio of 1/4. At times t such that $\tau_{\text{solid}} > t > \tau_{\text{dash}} > 0$, i.e. much greater than the relaxation time τ_{dash} of the dash line ribs, but much less than the relaxation time τ_{solid} of the solid line ribs, the honeycomb has a Poisson's ratio -1 corresponding to the re-entrant solid rib structure. In this regime, Poisson's ratio decreases with time. For greater times $t > \tau_{\text{solid}}$, all the honeycomb ribs have undergone significant relaxation, and the remaining material is essentially compliant foam in the interstices. The Poisson's ratio is that of the foam, $\nu = 0.3$. The time-dependent Poisson's ratio for such a model may be written for particular time constants ($\tau_{\text{dash}} = 0.01$ s, and $\tau_{\text{solid}} = 10$ s) as $\nu(t) = 0.25–1.25\,[1 - \exp(-t/0.01)] + 1.3\,[1 - \exp(-t/10)]$; the behavior is shown in Figure 3. Since the Poisson's ratio has decreased, then increased, it is not monotonic in time, contrary to the suggestion in [6].

Three-dimensional systems which exhibit a decreasing Poisson's ratio are as follows. Firstly, envisage a negative Poisson's ratio elastic foam [20] skeleton containing a microcellular viscoelastic foam with a conventional cell structure in the interstices [21]. For short times, this microcellular viscoelastic foam is designed to be sufficiently stiff that it provides most of the stiffness of the composite. Therefore the short-time Poisson's ratio approximates that of a conventional foam [15]: About 1/3. For long times, we assume that the modulus of the microcellular viscoelastic foam relaxes to

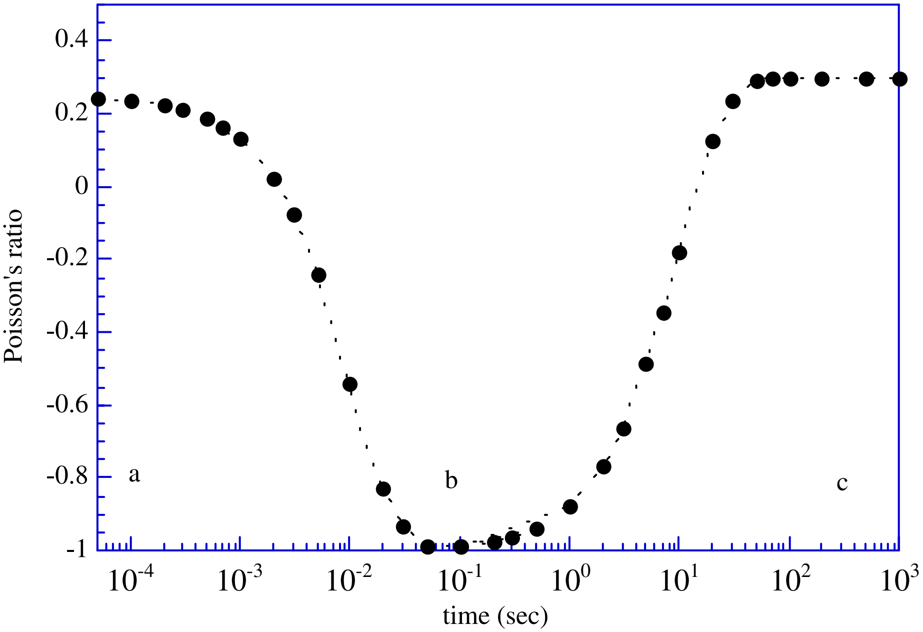

Figure 3 Time dependent Poisson's ratio for structure in Figure 2, for particular time constants for relaxation of ribs: $\tau_{\mathrm{dash}} = 0.01$ s for the *dash line ribs*, and $\tau_{\mathrm{solid}} = 10$ s for *solid line ribs*.

zero. Then the composite becomes equivalent to the re-entrant foam skeleton, which can have a Poisson's ratio as small as -0.7. Therefore, the Poisson's ratio decreases with time as in the two-dimensional case.

Secondly, envisage a porous material with a viscous fluid in the interstices. Stress-induced fluid flow gives rise to time-dependent behavior as analyzed by Biot [22]. This viscoelasticity depends on a volume change to move the fluid; shape changes in shear give rise to no viscoelasticity due to fluid flow. Since the effective bulk modulus of the fluid-filled sponge decreases with time, the Poisson's ratio also must decrease with time.

Thirdly, consider thermoelastic damping following Zener [23]. In any material which exhibits thermal expansion, the adiabatic compliance $S^{\mathrm{S}}_{\mathrm{ijkl}}$differs [24] from the isothermal compliance $S^{\mathrm{T}}_{\mathrm{ijkl}}$.

$$S^{\mathrm{S}}_{\mathrm{ijkl}} - S^{\mathrm{T}}_{\mathrm{ijkl}} = -\alpha_{\mathrm{ij}}\alpha_{\mathrm{kl}}\frac{T}{C^{\sigma}}, \tag{85}$$

with α_{ij} as the coefficient of thermal expansion, T as the absolute temperature and C^{σ} as the heat capacity at constant stress. For processes which are neither very fast nor very slow, stress-induced heat flow gives rise to dissipation of mechanical energy, observed as viscoelasticity. As with the above fluid-filled sponge, thermoelastic effects require a volume change. For example, thermoelastic relaxation in the shear modulus S_{2323} is zero, but Young's modulus E has a relaxation strength (defined above, (23)) $\Delta = \frac{\alpha^2 T}{C_v J^s}$, with α as the coefficient of thermal expansion, $J^s = S_{1111}$ as the adiabatic compliance $1/E$, T as the absolute temperature and C_v as the heat capacity at constant volume. These effects are usually small, e.g. for aluminum the relaxation strength is $\Delta = 0.0046$. Even so, in metals which exhibit small viscoelastic loss due to other causes, thermoelastic effects can be responsible for almost all the observed viscoelastic loss [25]. Since relaxation due to this cause occurs in bulk but not shear deformation, Poisson's ratio decreases with time in such materials.

6. Discussion

Poisson's ratio $\nu(t)$ need not increase with time and it need not be monotonic in time. This result is in contrast with that of Tschoegl et al. [6] who suggest that such a monotonic increase is required by the theory of viscoelasticity. Their analysis is based on writing $\nu(t)$ as a superposition of delay time terms, by analogy to the distribution of relaxation times used to describe the modulus. One can write a modulus as a distribution of exponential terms since in a passive material there is no internal source of energy, therefore the relaxation modulus must be monotonically decreasing. The proof [26] is based on analysis of an energy integral involving stress and strain. Since Poisson's ratio is a ratio of two strains, a corresponding energy integral is not physically meaningful, therefore one cannot conduct an analogous proof for Poisson's ratio. For the systems considered which exhibit increasing or non-monotonic Poisson's ratio, $\nu(t)$ cannot be written as a superposition of exponential terms of the same sign.

As for potential applications of materials with designed time dependent Poisson's ratio, one may envisage fasteners held by interference fit in which the fastening force increases with time or disappears entirely.

Poisson's ratio in relaxation and in creep differ, but for small to moderate relaxation strength, the difference is small. For most purposes, the error in ignoring the distinction between creep and relaxation Poisson's ratio is minimal.

7. Conclusions

The viscoelastic Poisson's ratio has a different time dependence depending on the test modality. Some may therefore question the role of Poisson's ratio as a material function in viscoelasticity. However, the difference between Poisson's ratios in creep and in relaxation is minor unless there is a large relaxation strength, as in the glass–rubber transition in a polymer. Correspondence principles are developed for relaxation type Poisson's ratio in the time domain, and complex Poisson's ratio in the frequency domain. The viscoelastic Poisson's ratio need not increase with time, and it need not be monotonic with time as is shown for selected material systems and in materials with designed microstructure.

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