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International Journal of Mechanical Sciences 48 (2006) 662–673
International Journal of MECHANICAL SCIENCES
www.elsevier.com/locate/ijmecsci

Generalized solution for predicting relaxation from creep in soft tissue: Application to ligament

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Received 23 February 2004; received in revised form 20 November 2005; accepted 20 December 2005
Available online 9 March 2006

Abstract

Creep and relaxation are two viscoelastic phenomena that are easily interrelated for a linearly viscoelastic material, but interrelationships are complex for nonlinearly viscoelastic materials. We use a single-integral nonlinear superposition principle to relate creep and relaxation, where the kernel is assumed to be a nonseparable product of strain and time. Herein, we develop time dependence as general power laws with up to four terms for creep compliance and relaxation modulus. Higher-order formulations give better results for ligament in terms of curve fitting and prediction of relaxation from creep. This is illustrated by a comparison between a two- and a three-term formulation on the experimental data of rabbit medial collateral ligaments. Also, an interrelation between several aspects of creep and relaxation is presented for arbitrarily high order, and the nature of high-order interrelation is discussed. The generality of the method makes it suitable to phenomenologically model many complex materials, to predict complex behaviors and to therefore reduce the amount of testing for robust material characterization.

Keywords: Ligament; Creep; Relaxation; Viscoelasticity

1. Introduction

Biological materials are complex microstructurally and complex in their mechanical behaviors. To describe them with a single robust model is a formidable challenge. An example of these complexities can be seen in the medial collateral ligament (MCL), often studied experimentally. It is present on the medial side of the knee and extends from the proximal tibia to the distal femur. Like other ligaments in the body, it is made up of collagen fibers, elastin and proteoglycans, glycolipids, water and cells. Collagen fibers dominate the microstructure and these fibers have a distinctive crimp pattern [1], which means that they have a wavy appearance. More and more fibrils are recruited with increasing tension to resist the tensile stress. Increased stretch straightens the initially crimped fibers so that they carry an increasing amount of load. This gives rise to a strain stiffening nonlinearity [2]. The nonlinearity indirectly provides a mechanism for relaxation to proceed faster than creep, as discussed by Lakes and Vanderby [3].

A number of microstructural [4–9], phenomenological [1,3,10,11] and continuum models [12–14] have been developed to describe ligament and tendon viscoelastic behavior. Few attempts have been made to critically test constitutive models. For example, Thornton et al. [9] found that creep and relaxation could not be adequately interrelated using a quasi-linear approach. Most of the interrelations in the nonlinear viscoelasticity literature are not based on superposition. They are discussed in detail in Oza et al. [11] and are briefly mentioned in this paper. Ashby and Jones [15] and Popov [16] analyze secondary creep but ignore primary creep, which is of interest here. Moreover, they assume the same relation between stress and strain rate in both creep and relaxation. Arutyunyan

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0020-7403/$ - see front matter © 2006 Published by Elsevier Ltd.
doi:10.1016/j.ijmecsci.2005.12.013

Nomenclature

t	time
τ	time variable of integration
$\sigma(t)$	time-dependent stress
σ	constant stress
$\varepsilon(t)$	time-dependent strain
ε	constant strain
$E(t, \varepsilon(\tau))$	relaxation modulus dependent on time and strain
$J(t, \sigma(\tau))$	creep compliance dependent on time and stress
$H(t)$	heaviside step function
g_d	coefficients in creep compliance kernel
f_d	coefficients in relaxation modulus kernel
n, m, p, r, k, i, j	powers of time in the creep compliance kernel
n, x, q, s, h	powers of time in the relaxation modulus kernel
a	power of stress and strain in creep compliance and relaxation modulus, respectively
u	any integer starting from 0, e.g. 0, 1, 2, 3…
d	any integer starting from 1, e.g. 1, 2, 3…
J	any integer starting from −1, e.g. −1, 0, 1, 2, 3…
Γ	gamma function
Σ	summation notation
X	symbol for limit used in the summation notation, which represents any integer starting from 1

[17] and Touati and Cederbaum [18] use a complex method to predict stress relaxation from creep; however, they do not directly use superposition methods. Fung's [10] quasi-linear viscoelasticity (QLV) assumes a separable kernel as a product of a function of time and a function of strain. QLV has been applied to many other soft tissues such as muscle, cartilage and brain tissue. It does not admit interrelations in which both creep and relaxation are represented in QLV form since a separable form for creep becomes a nonseparable form for relaxation as shown by Lakes and Vanderby [3]. QLV also fails to model properly the low load region where creep rate depends strongly on stress level [19]. Most earlier experiments involved only a single creep or relaxation test combined with a tensile stress strain curve to failure. This modality is insufficient to distinguish QLV from nonseparable nonlinear superposition. Multiple integral formulations developed by Lai and Findley [20] are not only more complicated but also more versatile than single integral forms.

Rheologic testing of complex biological materials (such as ligament) is inherently time-intensive. It would be desirable to perform creep tests, and then able to predict stress relaxation through a constitutive model so that the number of experiments required to study viscoelastic behavior can be halved. An interrelationship between creep and relaxation is then required. For a linearly viscoelastic material, such an interrelationship is simple and can be formulated using Laplace transformation of the constitutive equations. However, biological materials (e.g. the MCL) are nonlinear and require a more complex interrelationship. Thornton et al. [21] observed that relaxation proceeded faster than creep in ligaments and showed that linear viscoelastic theory was not able to phenomenologically model both behaviors with interrelated coefficients. They have shown that the predictions based on Laplace transformation (linear viscoelasticity) are poor. Hence, use of a nonlinear model to interrelate creep and relaxation is essential.

The following are single integral constitutive equations based on nonlinear superposition in which the relaxation function $E(t,\varepsilon)$ depends on strain ε and the creep function $J(t,\sigma)$ depends on stress σ. J is the ratio of time-dependent strain to constant (step) applied stress.

$$\sigma(t) = \int_0^t E(t-\tau, \varepsilon(\tau)) \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} \, \mathrm{d}\tau, \tag{1a}$$

$$\sigma(t) = \int_0^\tau J(t-\tau, \sigma(\tau)) \frac{\mathrm{d}\sigma}{\mathrm{d}\tau} \, \mathrm{d}\tau. \tag{1b}$$

The QLV model assumes the creep compliance $J(t,\sigma)$ to be separable into a product of time- and stress-dependent parts and is therefore a special case of the above equations. We do not make any QLV assumption here: Eqs. (1) are used without such restrictions. Recent studies by Provenzano et al. [22] demonstrated that the nonlinear superposition model can adequately model the strain-dependent stress-relaxation behavior of ligaments. They show that nonlinear superposition allows a more direct interpretation of the relationship between model parameters and physical behavior. Unlike QLV, the slope or the shape of the relaxation curve, not just its magnitude, can depend on strain.

The goal of this study is to formulate an interrelationship for general nonlinear superposition formulations for creep and relaxation and to show its ability to predict nonlinear relaxation from creep by using MCL data.

2. Interrelation of creep and relaxation

Time-dependent (creep) strain due to constant stress can be written as a sum of immediate and delayed Heaviside step functions in time $H(t)$

$$\varepsilon(t) = \varepsilon(0)H(t) + \sum_{i=0}^{N} \Delta\varepsilon_i H(t-t_i). \tag{2}$$

Each step strain in the summation gives rise to a relaxing component of stress in view of the definition of the relaxation function $E(t,\varepsilon)$, which is the ratio of stress to (step) strain ε as a function of time t and (in nonlinear materials) also of strain. Nonlinearity is accommodated in this analysis since the relaxation function E explicitly depends on strain level.

$$\varepsilon(t) = \varepsilon(0)E(t, \varepsilon(0)) + \sum_{i=0}^{N} \Delta\varepsilon_i E(t - t_i, \varepsilon(t_i)). \tag{3}$$

Here, we assume there is no effect from interactions between the step components, hence we consider single-integral-type nonlinear response [3] and exclude responses that require for their description a multiple integral formulation.

Dividing Eq. (3) by σ and using the definition of creep compliance $J(t,\sigma)$ yields

$$1 = J(0, \sigma)E(t, \varepsilon(0)) + \sum_{i=0}^{N} \Delta J_i E(t - t_i, \varepsilon(t_i)). \tag{4}$$

The creep compliance $J(t,\sigma)$ is the ratio of strain to (step) stress σ as a function of time t and (in nonlinear materials) also of stress. Pass to the limit of infinitely many fine step components to obtain a Stieltjes integral, with τ as a time variable of integration

$$1 = J(0, \sigma)E(t, \varepsilon(0)) + \int_0^t E(t - \tau, \varepsilon(\tau)) \frac{\partial J(\tau, \sigma)}{\partial \tau} \, d\tau. \tag{5}$$

As in the linear interrelation, the time dependence appears in the integral as dependence on a time variable of integration. Since for creep under step stress σ, $\sigma(t) = 0$ for $t<0$ and $\sigma(t) = \sigma$, i.e. time independent for $t>0$, we have $\varepsilon(t) = \sigma J(t,\sigma)$; so Eq. (5) becomes

$$1 = J(0, \sigma)E(t, \sigma J(0, \sigma)) + \int_0^t E(t - \tau, \sigma J(\tau, \sigma)) \frac{\partial J(\tau, \sigma)}{\partial \tau} \, d\tau. \tag{6}$$

To develop an explicit relationship between creep and relaxation, one assumes a particular functional form for one of the viscoelastic functions. For example, Lakes and Vanderby [3] used this Stieltjes integral to show that a separable form (QLV) of creep gives rise to a nonseparable relaxation function.

To obtain explicit interrelations, several explicit time-dependent functions are assumed. In the following, various nonseparable creep functions involving power laws in time are considered for primary creep. Creep in physiological ranges of loading is primary creep. Secondary creep involves irreversible damage and is outside the scope of this work. Power laws are used since they are suitable for modeling the behavior of materials of interest. Power-law terms have the limitation that the modulus tends to infinity as time tends to zero, an unrealistic situation. Since experimental data used for comparison are available over a limited window of the time domain, this asymptotic behavior is not obtrusive. A four-term nonlinear formulation is developed below. On the basis of an observed trend in the time dependencies for different stress dependencies, a more general solution is stated. As the number of terms increases, it becomes easier to obtain closer curve fits and interrelations, but the complexity increases as well.

2.1. Nonlinear formulation

The goals of this nonlinear formulation are to

(i) develop an interrelation for the first four terms in creep compliance and relaxation modulus,
(ii) illustrate the interrelation with experimental results for ligament, and
(iii) prove that the difference in the power of time of the dth term in creep and $(d-1)$th term is constant and is the same as that for the lower-order terms. This constrains the structure of any general form for interrelation of creep compliance and relaxation modulus, as discussed in Section 2.1.2.

2.1.1. Interrelation of the first four terms

A semi-inverse approach is used. The first term embodies the well-known power law in time within linear viscoelasticity. Higher-order terms are constructed in such a way as to fulfill the interrelation integral in Eq. (6).

Assume the creep behavior to be a series of powers of time t, as follows:

$$J(t, \sigma) = g_1 t^n + g_2 \sigma^a t^m + g_3 \sigma^{2a} t^p + g_4 \sigma^{3a} t^r + \cdots + g_{d-2} \sigma^{(u-2)a} t^k + g_{d-1} \sigma^{(u-1)a} t^i + g_d \sigma^{ua} t^j. \tag{6a}$$

The g, n, p and other coefficients are to be obtained by curve fitting from experimental creep data. The power terms n, m and p govern the slope of the creep curve; they are generally less than 1.

We assume a nonseparable power law form of relaxation modulus $E(t,\varepsilon)$, given as

$$E(t, \varepsilon) = f_1 t^{-n} + f_2 \varepsilon(t)^a t^{-x} + f_3 \varepsilon(t)^{2a} t^{-q} + f_4 \varepsilon(t)^{3a} t^{-s} + \cdots + f_d \varepsilon(t)^{ua} t^{-h}. \tag{6b}$$

The f coefficients are calculated in the following from the creep, assuming a single integral constitutive equations (1). Powers x, q and s are also calculated. The above two series are general ones and are developed from our earlier work on lower-order interrelations [11].

Eqs. (6a) and (6b) are assumed and an interrelationship is developed for the first four terms. The single-integral form based on a Stieltjes integral given as Eq. (5) is repeated here

$$1 = J(0, \sigma)E(t, \varepsilon(0)) + \int_0^t E(t - \tau, \varepsilon(\tau)) \frac{\partial J(\tau, \sigma)}{\partial \tau} \, d\tau. \tag{7}$$

The interrelation for the first four coefficients is as follows. Details of the analysis are given in Appendix 1.

$$1 = f_1 g_1 \frac{1}{\sin n\pi} n\pi, \tag{8}$$

$$f_2 = \frac{-mf_1 g_2 \Gamma(-n+1)\Gamma(m)}{n g_1^{a+1} \Gamma(m-2n+1)\Gamma(n)}, \tag{9}$$

$$\begin{aligned} f_3 = {} & \frac{1}{g_1^{(2a+1)} n\Gamma(n)\Gamma(2m-3n+1)} \\ & \times (-f_1 g_3 (2m-n)\Gamma(-n+1)\Gamma(2m-n) \\ & - f_2 g_2 g_1^a m\Gamma(m)\Gamma(m-2n+1) \\ & - \frac{\Gamma(a+1)}{\Gamma(a)} f_2 g_2 g_1^a n\Gamma(n)\Gamma(2m-3n+1)), \end{aligned} \tag{10}$$

$$\begin{aligned} f_4 = {} & \left\{ \frac{1}{n g_1^{(3a+1)} \Gamma(3m-4n+1)\Gamma(n)} \right\} \\ & \times [-(3m-2n) f_1 g_4 \{\Gamma(-n+1)\Gamma(3m-2n)\} \\ & - \frac{\Gamma(a+1)}{\Gamma(a)} n f_2 g_1 g_3 \{\Gamma(3m-4n+1)\Gamma(n)\} \\ & - \frac{\Gamma(2a+1)}{\Gamma(2a)} n g_1^{2a} g_2 f_3 \{\Gamma(3m-4n+1)\Gamma(n)\} \\ & - (2m-n) f_2 g_1^a g_3 \{\Gamma(m-2n+1)\Gamma(2m-n)\} \\ & - \frac{\Gamma(a+1)}{\Gamma(a)} m f_2 g_2^2 g_1^{a-1} \{\Gamma(2m-3n+1)\Gamma(m)\} \\ & - m f_3 g_1^{2a} g_2 \{\Gamma(-3n+2m+1)\Gamma(m)\} \\ & - \frac{\Gamma(a+1)}{\Gamma(a-1)} n f_2 g_2^2 g_1^{a-1} \{\Gamma(3m-4n+1)\Gamma(n)\}]. \end{aligned} \tag{11}$$

This achieves the first goal to develop an interrelation for the first four terms in creep compliance and relaxation modulus. Eqs. (8)–(11) give the interrelationships for f_1, f_2, f_3 and f_4, respectively, in the relaxation modulus $E(t,\varepsilon)$ in terms of the quantities in the creep compliance $J(t,\sigma)$.

2.1.2. *Proof by method of induction*

The increment in the power of the time term in creep compliance is by a fixed number, compared to the previous term. Our second goal was to prove that the difference in the power of time of the dth term in creep and the $(d-1)$th term is constant and is the same as that for the lower-order terms. The purpose is to generalize the low-order interrelation to many terms. From the above equations, we have seen that f_1 depends only on g_1, f_2 depends on g_1 and g_2, f_3 depends on g_1, g_2 and g_3 and f_4 depends on g_1, g_2, g_3 and g_4. So, f_d must depend on all the terms from g_1 to g_d, therefore high-order interrelations grow rapidly in complexity.

Analysis in Appendix 2 shows that the increment in the power of the time term in creep compliance is by a fixed number, compared to the previous term. It also demonstrates that creep compliance can be written in summation notation as follows:

$$J(t,\sigma) = \sum_{J=-1}^{X-2} \sum_{u=0}^{X-1} \sum_{d=1}^{X} g_d \sigma^{ua} t^{m+J_y}, \tag{12}$$

where X is any integer starting from 1, and $y = m-n$ from Appendix 2.

Rewriting Eq. (12) in an expanded form gives

$$J(t,\sigma) = g_1 t^n + g_2 \sigma^a t^m + g_3 \sigma^{2a} t^{2m-n} + g_4 \sigma^{3a} t^{3m-2n} + \cdots.$$

From Eqs. (30) and (34) in Appendix 1, $p = 2m-n$ and $r = 3m-2n$.

The above equation reduces to the following equation, which is the same as Eq. (6a).

$$J(t,\sigma) = g_1 t^n + g_2 \sigma^a t^m + g_3 \sigma^{2a} t^p + g_4 \sigma^{3a} t^r + \cdots.$$

Correspondingly the relaxation modulus in summation notation can be written as follows:

$$E(t,\varepsilon(t)) = \sum_{J=-1}^{X-2} \sum_{u=0}^{X-1} \sum_{d=1}^{X} f_d \varepsilon(t)^{ua} t^{-[(ua+3)n-2m-(J-1)y]}, \tag{13}$$

where X is any integer starting from 1, and $y = m-n$ from Appendix 2.

Rewriting Eq. (13) in an expanded form gives

$$\begin{aligned} E(t,\varepsilon) = {} & f_1 t^{-n} + f_2 \varepsilon(t)^a t^{-[(a+2)n-m]} + f_3 \varepsilon(t)^{2a} t^{-[(2a+3)n-2m]} \\ & + f_4 \varepsilon(t)^{3a} t^{-[(3a+4)n-3m]} + \cdots. \end{aligned}$$

From Eqs. (25), (29) and (34) in Appendix 1, $x = [(a+2)n-m]$, $q = [(2a+3)n-2m]$ and $s = [(3a+4)n-3m]$.

The above equation reduces to the following equation which is the same as Eq. (6b).

$$E(t,\varepsilon) = f_1 t^{-n} + f_2 \varepsilon(t)^a t^{-x} + f_3 \varepsilon(t)^{2a} t^{-q} + f_4 \varepsilon(t)^{3a} t^{-s} + \cdots.$$

Parameters obtained from experimental curve fits and theoretical predictions for two- and three-term equations are summarized in the Table 1.

The general form is illustrated in the following example. If we require the first three terms of creep and corresponding terms of relaxation, then substituting the result $y = m-n$ and $X = 3$ in Eq. (12) yields

$$J(t,\sigma) = g_1 t^{m-(m-n)} + g_2 \sigma^a t^m + g_3 \sigma^{2a} t^{m+(m-n)}.$$

So

$$J(t,\sigma) = g_1 t^n + g_2 \sigma^a t^m + g_3 \sigma^{2a} t^{2m-n}. \tag{14}$$

Table 1
Parameters obtained from experimental curve fits and theoretical predictions for two- and three-term equations

Model	Parameters (experimental fit)	Parameters (prediction)
Two-term	g_1, g_2, a, n, m	f_1, f_2
Three-term	g_1, g_2, g_3, a, n, m	f_1, f_2, f_3

Substituting $y = m - n$ and $X = 3$ in Eq. (13) yields

$$E(t, \varepsilon(t)) = f_1 t^{-[3n-2m+2(m-n)]} + f_2 \varepsilon(t)^a t^{-[(a+3)n-2m+m-n]} + f_3 \varepsilon(t)^{2a} t^{-[(2a+3)n-2m]}.$$

So

$$E(t, \varepsilon(t)) = f_1 t^{-n} + f_2 \varepsilon(t)^a t^{-[(2+a)n-m]} + f_3 \varepsilon(t)^{2a} t^{-[(2a+3)n-2m]}. \qquad (15)$$

These results match with those obtained from the above four-term formulation.

3. Application of the nonlinear formulation

The above method to model nonlinear creep and interrelate it to nonlinear stress relaxation are demonstrated now on rabbit MCL data. Both two- and three-term forms were used from the above general formulation to be applied to primary creep data for MCL of a rabbit.

Ligament data were collected for both creep and relaxation at different stress and strain levels, respectively [23]. Rabbit MCLs were serially tested to various levels of creep and relaxation for a period of 100 s followed by a period of recovery, which lasted 1000 s. The stress-relaxation tests were carried out first and corresponding creep tests followed on the contralateral ligament. Load values corresponding to the peak force of the contralateral stress relaxation test were used in the creep tests. The rationale for using contralateral ligaments was to avoid the possibility of nonrecoverable load history effects. The first loading was considered a preconditioning cycle and subsequent loadings were recorded. A preload of 0.5 N was applied to the ligament. Ligaments were kept saturated at ambient temperature. Deformation was inferred from digitized video images of small silicone markers on the ligament surface.

Ligament data were plotted on a log–log scale with the first point plotted at 1 s into the test. The rise time in the stress was 0.1 s. The creep rate was seen to decrease with higher levels of stress and the rate of relaxation was seen to decrease at higher levels of strain, which is due to the strain-stiffening nonlinearity observed in a ligament. Stresses at which three creep tests were carried out were 8.4, 18.4 and 32.8 MPa. Strains at which corresponding relaxation tests were carried out were 2.1%, 3.6% and 5.5%. It was assumed that there is no relation between the steps, so a multiple-integral form was not considered and instead a single-integral form was used.

The method used to fit the curves was as follows. The time scale of the creep and relaxation was divided by a constant (1.5 s for the ligament data) in order to simplify calculation of powers. Isochronal plots (strain vs. stress at a given time) were generated for two different times, 1.5 and 90 s). The first isochronal was curve fitted with $\varepsilon = g_1\sigma + g_2\sigma^{1.85}$ for a two-term form to obtain the values for g_1 and g_2, and $\varepsilon = g_1\sigma + g_2\sigma^{1.85} + g_2\sigma^{2.7}$ for a three-term

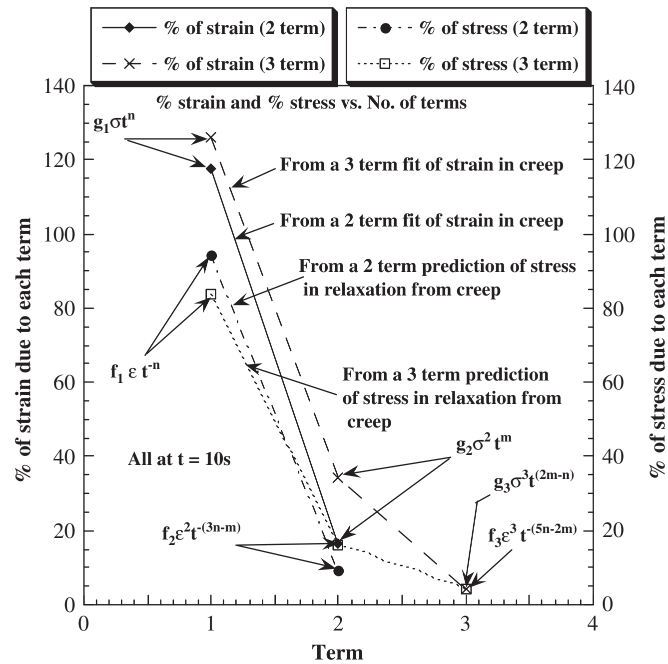

Fig. 1. This graph shows the percent strain due to each term in the two- and three-term curve fit for creep given by $\varepsilon(t) = g_1\sigma t^n + g_2\sigma^2 t^m$ and $\varepsilon(t) = g_1\sigma t^n + g_2\sigma^2 t^m + g_3\sigma^3 t^{2m-n}$, respectively, and the percent stress due to each term in the two- and three-term prediction for relaxation given by $\sigma(t) = f_1\varepsilon t^{-n} + f_2\varepsilon^2 t^{-(3n-m)}$ and $\sigma(t) = f_1\varepsilon t^{-n} + f_2\varepsilon^2 t^{-(3n-m)} + f_3\sigma^3 t^{-(5n-2m)}$, respectively. g_1, g_2, g_3, n and m are obtained by fitting the isochronals obtained from creep, while f_1 and f_2 are obtained from the interrelation. Values of each term in the graph are for $t = 10$ s. In these interrelations, g_2 is negative. All the values are represented as modulus of % strain.

form to obtain the values for g_1, g_2 and g_3. The second isochronal was curve fitted with $\varepsilon = g_1\sigma t^n + g_2\sigma^{1.85} t^m$ for a two-term form with known values of g_1 and g_2 to obtain n and m. A three-term fit based on $\varepsilon = g_1\sigma t^n + g_2\sigma^{1.85} t^m + g_2\sigma^{2.7} t^{2m-n}$ with known values of g_1, g_2 and g_3 was used to obtain n and m. Owing to the relation $\varepsilon = \sigma * J(t,\sigma)$, 1.85 from the above equation refers to '$a+1$' from Eq. (6a), 2.7 refers to '$2a+1$' from Eq. (6a) and '$2m-n$' refers to 'p' from Eq. (6a). These values of g_1, g_2, g_3, n and m were used to fit the different creep curves. Kaleidagraph (Synergy Software, 2457 Perkiomen Avenue, Reading, PA 19606, USA) uses the nonlinear least-squares estimation based on the Levenberg–Marquardt algorithm [24] for curve fitting.

It is shown in Fig. 1 that the percentage of strain and stress due to the first term in creep ($g_1\sigma t^n$) and relaxation ($f_1\varepsilon t^{-n}$), respectively, was higher than that due to the rest of the terms. Also, the percentage of strain and stress in the corresponding terms becomes less. The first isochronal was curve fitted with $\varepsilon = g_1\sigma + g_2\sigma^{1.85}$ for a two-term form to obtain the values for g_1 and g_2. This process yields a very high value of g_1 and a negative value of g_2. This is the reason for the term $g_1\sigma t^n$ to be more than 100%. It is also to be noted that absolute values of g_1 and g_2 are used in the Fig. 1. Percentage stress and strain due to each term is calculated as follows:

Percentage strain due to first term in creep

$$= \frac{|g_1|t^n}{(|g_1|\sigma(t)t^n + |g_2|\sigma^{a+1}t^m + \cdots)}.$$

Percentage stress due to first term in relaxation

$$= \frac{|f_1|t^{-n}}{(|f_1|\varepsilon(t)t^{-n} + |f_2|\varepsilon(t)^{a+1}t^{-x} + \cdots)},$$

where $|g_1|$, $|g_2|$, $|f_1|$ and $|f_2|$ represent the absolute observed values.

A four-term form cannot be used to fit isochronals obtained from three creep curves since there are too many parameters to fit the number of points. So, if there are 'n' creep curves we are trying to fit then the number of terms in the creep compliance can either be equal to 'n' or less than 'n'.

Using the interrelation in Eqs. (8)–(10), the predicted relaxation curve was generated and is shown in Fig. 3. It is seen that a three-term prediction is much better than a two-term prediction within the range of interpolated data. The window of time and strain in these experimental results is relatively narrow. Results over a wider range would likely be more demanding, and could then require more terms for fitting and for interrelation.

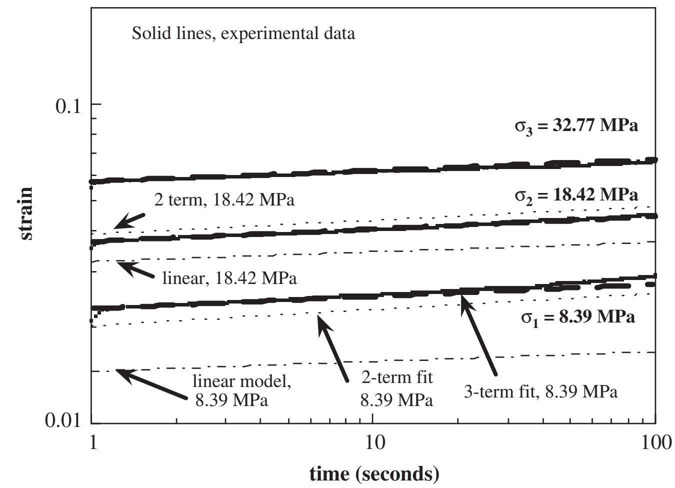

Fig. 2. Curve fitting of creep of ligament. Data of rabbit medial collateral ligament at three different stress levels, σ_1 (8.4 MPa), σ_2 (18.4 MPa) and σ_3 (32.8 MPa). g_1, g_2, g_3, n and m are obtained by fitting the isochronals obtained from creep. Two- and three-term fits for creep are given by $\varepsilon(t) = g_1\sigma t^n + g_2\sigma^{1.85}t^m$ and $\varepsilon(t) = g_1\sigma t^n + g_2\sigma^{1.85}t^m + g_3\sigma^{2.7}t^{2m-n}$, respectively. Linear fit for creep is given by $\varepsilon(t) = g_1\sigma t^n$. Dense points (solid line): experiment. Thick long dashes (————) are the three-term fits, thin short dashes (-----------) are the two-term fits and center lines (–·–·–·–·–) are the linear fits.

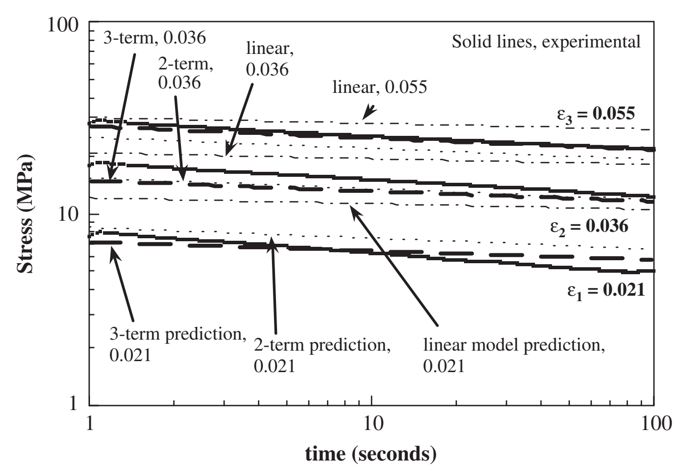

Fig. 3. Prediction of relaxation from creep; comparison with experimental relaxation of ligament. The three corresponding strain levels for relaxation are ε_1 (2.1%), ε_2 (3.6%) and ε_3 (5.5%). Two- and three-term predictions for relaxation are given by $\sigma(t) = f_1\varepsilon t^{-n} + f_2\varepsilon^{1.85}t^{-(2.85n-m)}$ and $\sigma(t) = f_1\varepsilon t^{-n} + f_2\varepsilon^{1.85}t^{-(2.85n-m)} + f_3\varepsilon^{2.7}t^{-(4.7n-2m)}$, respectively. Linear predictions for relaxation is given by $\sigma(t) = f_1\varepsilon t^{-n}$. Dense points (solid line): experiment. Thick long dashes (————) are the three-term predictions, thin short dashes (-----------) are the two-term predictions, center lines (–·–·–·–·–) are the linear predictions.

4. Discussion

The goal of this study is to formulate an interrelation for general nonlinear superposition formulations for creep and relaxation and to show its ability to predict nonlinear relaxation from creep, using MCL data. Creep and relaxation were interrelated using the single-integral nonlinear superposition principle. The model was fit to MCL data in which the rate of creep was stress dependent. The model was then used to predict stress relaxation. All predictions made are within the range of interpolated data and may not hold true for the extrapolated data. Also, if sprain damage occurs in a ligament, new physical processes occur. The fits and predictions obtained with higher number of terms are better than those obtained with a lesser number of terms. Better results with curve fitting and prediction are obtained with more number of terms in creep compliance and relaxation modulus. However, the number of terms in the creep compliance kernel cannot be more than the number of creep curves being fitted as the system would then be overdetermined. Figs. 2 and 3 show the fit and the prediction obtained with a two- and a three-term formulation. Lower-order terms accounted for most of the time dependence over this narrow range of time scale. Because of the generality in the phenomenological viscoelastic model and because of the analytical interrelationship developed herein, we anticipate the method can be used to model robustly many complex viscoelastic materials over larger ranges of time and strain. Limitations of power-law creep considered here include the singularity at time zero (which is not accessible experimentally), as well as the complexity of model implementations involving many terms. These, however, are more than compensated by the experimental test time and cost, which may be saved by using analytical interrelations. In addition, the model has limitations that are inherent with a single-integral nonlinear superposition approach to viscoelasticity. Further testing should be done to explore its applicability to more complex loading regimes.

Acknowledgment

The financial support from the NSF via Grant CMS 9907977 is gratefully acknowledged.

Appendix 1

This section contains the details of the analysis. Derivation of the four-term interrelation is as follows:

The derivative of the creep function $J(t)$ is

$$\frac{\mathrm{d}J(\tau,\sigma)}{\mathrm{d}\tau} = ng_1t^{n-1} + mg_2\sigma^a t^{m-1} + pg_3\sigma^{2a}t^{p-1} + rg_4\sigma^{3a}t^{r-1} + \cdots + kg_{d-2}\sigma^{(u-2)a}t^{k-1} + ig_{d-1}\sigma^{(u-1)a}t^{i-1} + jg_d\sigma^{ua}t^{j-1}. \tag{16}$$

Since

$$\varepsilon(t) = J(t)\sigma. \tag{17}$$

So

$$\varepsilon(t) = g_1\sigma t^n + g_2\sigma^{a+1}t^m + g_3\sigma^{2a+1}t^p + g_4\sigma^{3a+1}t^r + \cdots + g_{d-2}\sigma^{(f-2)a+1}t^k + g_{d-1}\sigma^{(f-1)a+1}t^i + g_d\sigma f^{a+1}t^j.$$

Substituting the above result of $\varepsilon(t)$ into the equation for the relaxation modulus, we get

$$\begin{aligned}E(t,\varepsilon(t)) =\ & f_1t^{-n} + f_2t^{-x}[g_1\sigma t^n + g_2\sigma^{a+1}t^m + g_3\sigma^{2a+1}t^p + g_4\sigma^{3a+1}t^r + \cdots + g_{d-2}\sigma^{(u-2)a+1}t^k \\ & + g_{d-1}\sigma^{(u-1)a+1}t^i + g_d\sigma^{ua+1}t^j]a \\ & + f_3t^{-q}[g_1\sigma t^n + g_2\sigma^{a+1}t^m + g_3\sigma^{2a+1}t^p + g_4\sigma^{3a+1}t^r + \cdots + g_{d-2}\sigma^{(u-2)a+1}t^k \\ & + g_{d-1}\sigma^{(u-1)a+1}t^i + g_d\sigma^{ua+1}t^j]^b + f_4t^{-s} \\ & \times [g_1\sigma t^n + g_2\sigma^{a+1}t^m + g_3\sigma^{2a+1}t^p + g_4\sigma^{3a+1}t^r + \cdots + g_{d-2}\sigma^{(u-2)a+1}t^k \\ & + g_{d-1}\sigma^{(u-1)a+1}t^i + g_d\sigma^{ua+1}t^j]^c \\ & + \cdots + f_d[g_1\sigma t^n + g_2\sigma^{a+1}t^m + g_3\sigma^{2a+1}t^p + g_4\sigma^{3a+1}t^r + \cdots + g_{d-2}\sigma^{(u-2)a+1}t^k \\ & + g_{d-1}\sigma^{(u-1)a+1}t^i + g_d\sigma^{ua+1}t^j]^{ua}t^{-h}.\end{aligned} \tag{18}$$

The interrelationship is developed for the first four terms. Since we are conducting a third-order analysis in σ, we ignore all the terms in Eq. (18) above the power of 3 in a.

The binomial expansion can be given as follows [21]:

$$(x+y)^\upsilon = \sum_{n=0}^{\infty}\frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-n+1)}\frac{x^n y^{\upsilon-n}}{n!}$$

and the gamma function is defined as

$$\Gamma(\upsilon) = \int_0^\infty t^{\upsilon-1}\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

Gamma functions appear in the equation of the relaxation modulus below since the binomial expansion involves the use of gamma functions.

$$\begin{aligned}E(t,\varepsilon(t)) =\ & f_1t^{-n} + f_2t^{-x}[g_1^a\sigma^a t^{an} + \frac{\Gamma(a+1)}{\Gamma(a)}g_2g_1^{a-1} \\ & \times\sigma^{2a}t^{m+(a-1)n} + \frac{\Gamma(a+1)}{\Gamma(a)}g_3g_1^{a-1}\sigma^{3a}t^{p+(a-1)n} \\ & + \cdots + \frac{\Gamma(a+1)}{\Gamma(a)}g_{d-1}g_1^{a-1}\sigma^{ua}t^{i+(a-1)n} \\ & + \frac{1}{2}\frac{\Gamma(a+1)}{\Gamma(a-1)}g_2^2g_1^{a-2}\sigma^{3a}t^{2m+(a-2)n} \\ & + \cdots + g_2g_{d-2}g_1^{a-2}\sigma^{ua}t^{m+k+(a-2)n}] + f^3t^{-q} \\ & \times[g_1^{2a}\sigma^{2a}t^{2an} + \frac{\Gamma(2a+1)}{\Gamma(2a)}g_2g_1^{2a-1}\sigma^{3a}t^{m+(2a-1)}n \\ & + \cdots + \frac{\Gamma(2a+1)}{\Gamma(2a)}g_{d-2}g_1^{2a-1}\sigma^{ua}t^{k+(2a-1)n}] \\ & + f_4t^{-s}[g_1^c\sigma^c t^{cn} + \ldots] + \cdots + f_dt^{-h}[g_1^{ua}\sigma^{ua}t^{uan}],\end{aligned}$$

$$\begin{aligned}E(t,\varepsilon(t)) =\ & f_1t^{-n} + f_2g_1^a\sigma^a t^{an-x} + \frac{\Gamma(a+1)}{\Gamma(a)}f_2g_2g_1^{a-1} \\ & \times\sigma^{2a}t^{m+(a-1)n-x} + \frac{\Gamma(a+1)}{\Gamma(a)}f_2g_3g_1^{a-1} \\ & \times\sigma^{3a}t^{p+(a-1)n-x} + \frac{\Gamma(a+1)}{\Gamma(a)}f_2g_{d-1}g_1^{a-1}\sigma^{ua}t^i \\ & + (a-1)^{n-x} + \frac{1}{2}\frac{\Gamma(a+1)}{\Gamma(a-1)}f_2g_2^2g_1^{a-2} \\ & \times\sigma^{3a}t^{2m+(a-2)n-x} + \cdots + f_2g_2g_{d-2}g_1^{a-2} \\ & \times\sigma^{ua}t^{m+k+(a-2)n-x} + f_3g_1^{2a}\sigma^{2a}t^{2an-q} \\ & + \frac{\Gamma(2a+1)}{\Gamma(2a)}f_3g_2g_1^{2a-1}\sigma^{3a}t^{m+(2a-1)n-q} \\ & + \cdots + \frac{\Gamma(2a+1)}{\Gamma(2a)}f_3g_{d-2}g_1^{2a-1}\sigma^{ua}t^{k+(2a-1)n-q} \\ & + f_4g_1^{3a}\sigma^{3a}t^{3an-s} + \cdots + f_dg_1^{ua}\sigma^{ua}t^{uan-h},\end{aligned} \tag{19}$$

$J(0) = 0$. Therefore, the first term in the Stieltjes integral vanishes.

Upon substituting Eqs. (19) and (16) into the Stieltjes integral given in Eq. (7) in Section 2.1.1 we obtain

$$\begin{aligned}1 = \int_0^t \{ & f_1t^{-n} + f_2g_1^a\sigma^a t^{an-x} + \frac{\Gamma(a+1)}{\Gamma(a)}f_2g_2g_1^{a-1}\sigma^{2a}t^{m+(a-1)n-x} \\ & + \frac{\Gamma(a+1)}{\Gamma(a)}f_2g_3g_1^{a-1}\sigma^{a3a}t^{p+(a-1)n-x} + \frac{\Gamma(a+1)}{\Gamma(a)}f_2g_{d-1}g_1^{a-1} \\ & \times\sigma^{ua}t^{i+(a-1)n-x} + \frac{1}{2}\frac{\Gamma(a+1)}{\Gamma(a-1)}f_2g_2^2g_1^{a-2}\sigma^{3a}t^{2m+(a-2)n-x} \\ & + \cdots + f_2g_2g_{d-2}g_1^{a-2}\sigma^{ua}t^{m+k+(a-2)n-x}\end{aligned}$$

$$+f_3g_1^{2a}\sigma^{2a}t^{2an-q}+\frac{\Gamma(2a+1)}{\Gamma(2a)}f_3g_2g_1^{2a-1}\sigma^{3a}t^{m+(2a-1)n-q}$$
$$+\cdots+\frac{\Gamma(2a+1)}{\Gamma(2a)}f_3g_{d-2}g_1^{2a-1}\sigma^{ua}t^{k+(2a-1)n-q}$$
$$+f_4g_1^{3a}\sigma^{3a}t^{3an-s}+\cdots+f_dg_1^{ua}\sigma^{ua}t^{uan-h}\}$$
$$\{ng_1\tau^{n-1}+mg_2\sigma^a\tau^{m-1}+pg_3\sigma^{2a}\tau^{p-1}$$
$$+rg_4\sigma^{3a}\tau^{r-1}+\dots\dots\dots+kg_{d-2}\sigma^{(u-2)a}t^{k-1}$$
$$+ig_{d-1}\sigma^{(u-1)a}t^{i-1}+jg_d\sigma^{ua}t^{j-1}\}\mathrm{d}\tau. \tag{20}$$

Eq. (20) is of the form

$$1+(0)\sigma+(0)\sigma^2+(0)\sigma^3=A+B\sigma+C\sigma^2+D\sigma^3. \tag{21}$$

So

$$A=1,\ B=0,\ C=0 \text{ and } D=0. \tag{22}$$

On the basis of Eqs. (21) and (22), we equate 'σ'-independent terms in Eq. (20) to 1 and all 'σ^a', 'σ^{2a}', 'σ^{3a}' and 'σ^{ua}' terms in Eq. (20) to 0

From Eq. (20), we get

$$1=f_1g_1\int_0^t n(t-\tau)^{-n}\tau^{n-1}\,\mathrm{d}\tau,$$
$$1=f_1g_1\frac{1}{\sin n\pi}n\pi. \tag{23}$$

The above equation is the same as Eq. (8) in Section 2.1.1. Now, we take all the 'σ^a' terms and equate them to 0. From Eq. (20), we get

$$0=f_1g_2\int_0^\tau m(t-\tau)^{-n}\tau^{m-1}\,\mathrm{d}\tau$$
$$+f_2g_1^{a+1}\int_0^\tau n(t-\tau)^{an-x}\tau^{n-1}\,\mathrm{d}\tau,$$
$$0=f_1g_2m\left\{\frac{\Gamma(-n+1)\Gamma(m)}{\Gamma(m-n+1)}\right\}t^{m-n}+f_2g_1^{a+1}n$$
$$\times\left\{\frac{\Gamma(an-x+1)\Gamma(n)}{\Gamma((a+1)n-x+1)}\right\}t^{(a+1)n-x}. \tag{24}$$

For Eq. (24) to be true, there are two solutions:

(1) f_1, f_2, g_1, g_2, m or n are zero or
(2) powers of time of both the terms are the same, i.e. $m{-}n=(a+1)n{-}x$.

So, since solution 1 entails an elastic not a viscoelastic material,

$$m-n=(a+1)n-x.$$

Therefore,

$$x=(2+a)n-m, \tag{25}$$

$$0=\left\{f_1g_2m\left\{\frac{\Gamma(-n+1)\Gamma(m)}{\Gamma(m-n+1)}\right\}t^{m-n}\right.$$
$$\left.+f_2g_1^{a+1}n\left\{\frac{\Gamma(an-x+1)\Gamma(n)}{\Gamma((a+1)n-x+1)}\right\}\right\}t^{m-n}.$$

Therefore

$$0=f_1g_2m\left\{\frac{\Gamma(-n+1)\Gamma(m)}{\Gamma(m-n+1)}\right\}$$
$$+f_2g_1^{a+1}n\left\{\frac{\Gamma(an-x+1)\Gamma(n)}{\Gamma((a+1)n-x+1)}\right\}.$$

The first solution is not of interest, since it corresponds to a purely elastic material, so the second one is used to account for the viscoelasticity.

Canceling out the common terms from Eq. (24), we get

$$0=f_1g_2\{(\Gamma(-n+1)\Gamma(m)\}$$
$$+f_2g_1^{a+1}\{\Gamma(m-2n+1)\Gamma(n)\}. \tag{26}$$

Solving for f_2, we obtain

$$f_2=\frac{-mf_1g_2\Gamma(-n+1)\Gamma(m)}{ng_1^{a+1}\Gamma(m-2n+1)\Gamma(n)}. \tag{27}$$

The above equation is the same as Eq. (9) in Section 2.1.1. Now, we equate all 'σ^{2a}' terms to 0. From Eq. (20), we get

$$0=f_1g_3\int_0^t p(t-\tau)^{-n}\tau^{p-1}\,\mathrm{d}\tau+f_2g_1^ag_2\int_0^t m(t-\tau)^{an-x}$$
$$\times\tau^{m-1}\mathrm{d}\tau+\frac{\Gamma(a+1)}{\Gamma(a)}f_2g_1^ag_2\int_0^t n(t-\tau)^{m+(a-1)n-x}$$
$$\times\tau^{n-1}\mathrm{d}\tau+f_3g_1^b\int_0^t n(t-\tau)^{2an-q}\tau^{n-1}\,\mathrm{d}\tau,$$
$$0=f_1g_3p\left\{\frac{\Gamma(-n+1)\Gamma(p)}{\Gamma(p-n+1)}\right\}t^{p-n}$$
$$+f_2g_1^ag_2m\left\{\frac{\Gamma(an-x+1)\Gamma(m)}{\Gamma(an-x+m+1)}\right\}t^{an-x+m}+\frac{\Gamma(a+1)}{\Gamma(a)}$$
$$\times f_2g_1^ag_2n\left\{\frac{\Gamma(m+(a-1)n-x+1)\Gamma(n)}{\Gamma(m+an-x+1)}\right\}t^{an-x+m}$$
$$+f_3g_1^{2a}n\left\{\frac{\Gamma(2an-q+1)\Gamma(n)}{\Gamma((2a+1)n-q+1)}\right\}t^{(2a+1)n-q}. \tag{28}$$

For Eq. (28) to be true, there are two solutions:

(1) $f_1, f_2, g_1, g_2, g_3, m$ or n are zero or
(2) powers of time of all the terms in Eq. (28) are the same, i.e. $p{-}n=an{-}x+m=(2a{+}1)n{-}q$.

So, $(2a+1)n{-}q=an{-}x+m=p{-}n$. Since $x=(2{+}a)n{-}m$, $(2a{+}1)n{-}q=an{-}2n{-}an{+}m{+}m=2m{-}2n$. So

$$q=(2a+3)n-2m, \tag{29}$$

$$2m-2n=p-n,$$
$$p=2m-n. \tag{30}$$

If powers of time terms are the same then Eq. (28) can be written as

$$0=\left\{f_3g_1^{2a}n\left\{\frac{\Gamma(2an-q+1)\Gamma(n)}{\Gamma((2a+1)n-q+1)}\right\}+f_1g_3p\left\{\frac{\Gamma(-n+1)\Gamma(p)}{\Gamma(p-n+1)}\right\}+f_2g_1^ag_2m\left\{\frac{\Gamma(an-x+1)\Gamma(m)}{\Gamma(an-x+m+1)}\right\}+\frac{\Gamma(a+1)}{\Gamma(a)}\times f_2g_1^ag_2n\left\{\frac{\Gamma(m+(a-1)n-x+1)\Gamma(n)}{\Gamma(m+an-x+1)}\right\}\right\}t^{p-n}.$$

So

$$0=f_3g_1^{2a}n\left\{\frac{\Gamma(2an-q+1)\Gamma(n)}{\Gamma((2a+1)n-q+1)}\right\}+f_1g_3p\left\{\frac{\Gamma(-n+1)\Gamma(p)}{\Gamma(p-n+1)}\right\}+f_2g_1^ag_2m\left\{\frac{\Gamma(an-x+1)\Gamma(m)}{\Gamma(an-x+m+1)}\right\}+\frac{\Gamma(a+1)}{\Gamma(a)}f_2g_1^ag_2n\left\{\frac{\Gamma(m+(a-1)n-x+1)\Gamma(n)}{\Gamma(m+an-x+1)}\right\}.$$

Again, only the second solution is of interest since ligament is a viscoelastic material.

Substituting results (30) and (29) into (28), we get

$$0=f_1g_3(2m-n)\left\{\frac{\Gamma(-n+1)\Gamma(2m-n)}{\Gamma(2m-2n+1)}\right\}t^{2(m-n)}+f_2g_1^ag_2m\left\{\frac{\Gamma(m-2n+1)\Gamma(m)}{\Gamma(2m-2n+1)}\right\}t^{2(m-n)}+\frac{\Gamma(a+1)}{\Gamma(a)}f_2g_1^ag_2n\left\{\frac{\Gamma(2m-3n+1)\Gamma(n)}{\Gamma(2m-2n+1)}\right\}t^{2(m-n)}+f_3g_1^{2a+1}n\left\{\frac{\Gamma(2m-3n+1)\Gamma(n)}{\Gamma(2m-2n+1)}\right\}t^{2(m-n)}. \quad (31)$$

Simplifying Eq. (31) and solving for f_3, we get

$$0=f_1g_3(2m-n)\{\Gamma(-n+1)\Gamma(2m-n)\}+f_2g_1^ag_2m\{\Gamma(m-2n+1)\Gamma(m)\}+\frac{\Gamma(a+1)}{\Gamma(a)}\times f_2g_1^ag_2n\{\Gamma(2m-3n+1)G(n)\}+f_3g_1^{2a+1}n\{\Gamma(2m-3n+1)G(n)\},$$

$$f_3=\frac{1}{g_1^{(2a+1)}n\Gamma(n)\Gamma(2m-3n+1)}\times(-f_1g_3(2m-n)\Gamma(-n+1)\Gamma(2m-n)-f_2g_2g_1^am\Gamma(m)\Gamma(m-2n+1)-\frac{\Gamma(a+1)}{\Gamma(a)}\times f_2g_2g_1^an\Gamma(n)\Gamma(2m-3n+1)). \quad (32)$$

The above equation is the same as Eq. (10) in Section 2.1.1.

Now we equate all 'σ^{3a}' terms to 0.

From Eq. (20), we get

$$0=\int_0^t rf_1g_4(t-\tau)^{-n}\tau^{r-1}\mathrm{d}\tau+\frac{\Gamma(a+1)}{\Gamma(a)}\int_0^t nf_2g_{1^a}g_3\times(t-\tau)^{p+(a-1)n-x}\tau^{n-1}\,\mathrm{d}\tau+\frac{\Gamma(2a+1)}{\Gamma(2a)}\int_0^t ng_1^{2a}g_2f_3\times(t-\tau)^{m+(2a-1)n-q}\tau^{n-1}\,\mathrm{d}\tau+\int_0^t nf_4g_1^{3a+1}(t-\tau)^{3an-s}\times\tau^{n-1}\,\mathrm{d}\tau+\int_0^t pf_2g_1^ag_3(t-\tau)^{an-x}\tau^{p-1}\mathrm{d}\tau+\frac{\Gamma(a+1)}{\Gamma(a)}\times\int_0^t mf_2g_{2^2}g_{1^{a-1}}(t-\tau)^{m+(a-1)n-x}\tau^{m-1}\,\mathrm{d}\tau+\int_0^t mf_3g_1^{2a}g_2\times(t-\tau)^{2an-q}\tau^{m-1}\,\mathrm{d}\tau+\frac{\Gamma(a+1)}{\Gamma(a-1)}\int_0^t nf_2g_{2^2}g_1^{a-1}\times(t-\tau)^{2m+(a-2)n-x}\tau^{n-1}\,\mathrm{d}\tau.$$

$$0=rf_1g_4\,\frac{\Gamma(-n+1)\Gamma(r)}{\Gamma(r-n+1)}\,t^{r-n}+\frac{\Gamma(a+1)}{\Gamma(a)}nf_2g_1^ag_3\times\frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)}\,t^{3m-3n}+\frac{\Gamma(2a+1)}{\Gamma(2a)}\,ng_1^{2a}g_2f_3\times\frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)}\,t^{3m-3n}+nf_4g_1^{3a+1}\times\frac{\Gamma(3an-s+1)\Gamma(n)}{\Gamma((3a+1)n-s+1)}\,t^{(3a+1)n-s}+(2m-n)f_2g_1^ag_3\times\frac{\Gamma(m-2n+1)\Gamma(2m-n)}{\Gamma(3m-3n+1)}\,t^{3m-3n}+\frac{\Gamma(a+1)}{\Gamma(a)}\,mf_2g_2^2g_1^{a-1}\times\frac{\Gamma(2m-3n+1)\Gamma(m)}{\Gamma(3m-3n+1)}\,t^{3m-3n}+mf_3g_1^{2a}g_2\times\frac{\Gamma(-3n+2m+1)\Gamma(m)}{\Gamma(3m-3n+1)}\,t^{3m-3n}+\frac{\Gamma(a+1)}{\Gamma(a-1)}\,nf_2g_2^2g_1^{a-1}\times\frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)}\,t^{3m-3n}. \quad (33)$$

For Eq. (33) to be true, there are two solutions:

(1) f_1, f_2, f_3, g_1, g_2, g_3, g_4, m or n are zero or
(2) powers of time of all the terms in Eq. (33) are the same, i.e. $r-n=3m-3n=(3a+1)n-s$.

So, $r-n=3m-3n$.

Thus,

$$r=3m-2n \quad (34)$$

and $(3a+1)n-s=3m-3n$.

So,

$$s=(3a+4)n-3m. \quad (35)$$

If powers of the time terms are the same, then Eq. (33) can be written as

$$0=\left\{rf_1g_4\,\frac{\Gamma(-n+1)\Gamma(r)}{\Gamma(r-n+1)}+\frac{\Gamma(a+1)}{\Gamma(a)}\,nf_2g_1^ag_3\times\frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)}+\frac{\Gamma(2a+1)}{\Gamma(2a)}ng_1^{2a}g_2f_3\right.$$

$$\times \frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)} + nf_4 g_1^{3a+1} \frac{\Gamma(3an-s+1)\Gamma(n)}{\Gamma((3a+1)n-s+1)} + (2m-n)f_2 g_1^a g_3 \frac{\Gamma(m-2n+1)\Gamma(2m-n)}{\Gamma(3m-3n+1)} + \frac{\Gamma(a+1)}{\Gamma(a)} mf_2 g_2^2 g_1^{a-1} + \frac{\Gamma(2m-3n+1)\Gamma(m)}{\Gamma(3m-3n+1)} + mf_3 g_1^{2a} g_2 \frac{\Gamma(-3n+2m+1)\Gamma(m)}{\Gamma(3m-3n+1)} + \frac{\Gamma(a+1)}{\Gamma(a-1)} nf_2 g_2^2 g_1^{a-1} \frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)} \Bigg\} t^{r-n}.$$

So,

$$0 = rf_1 g_4 \frac{\Gamma(-n+1)\Gamma(r)}{\Gamma(r-n+1)} + \frac{\Gamma(a+1)}{\Gamma(a)} nf_2 g_1^a g_3 \times \frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)} + \frac{\Gamma(2a+1)}{\Gamma(2a)} ng_1^{2a} g_2 f_3 \times \frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)} + nf_4 g_1^{3a+1} \frac{\Gamma(3an-s+1)\Gamma(n)}{\Gamma((3a+1)n-s+1)} + (2m-n)f_2 g_1^a g_3 \frac{\Gamma(m-2n+1)\Gamma(2m-n)}{\Gamma(3m-3n+1)} + \frac{\Gamma(a+1)}{\Gamma(a)} mf_2 g_2^2 g_1^{a-1} \frac{\Gamma(2m-3n+1)\Gamma(m)}{\Gamma(3m-3n+1)} + mf_3 g_1^{2a} g_2 \frac{\Gamma(-3n+2m+1)\Gamma(m)}{\Gamma(3m-3n+1)} + \frac{\Gamma(a+1)}{\Gamma(a-1)} nf_2 g_2^2 g_1^{a-1} \frac{\Gamma(3m-4n+1)\Gamma(n)}{\Gamma(3m-3n+1)}.$$

Again, only the second solution is of interest since ligament is a viscoelastic material.

Substituting Eqs. (34) and (35) into Eq. (33) and solving for f_4 gives us

$$f_4 = \left\{ \frac{1}{ng_1^{(3a+1)}\Gamma(3m-4n+1)\Gamma(n)} \right\} \Big[-(3m-2n)f_1 g_4 \times \{\Gamma(-n+1)\Gamma(3m-2n)\} - \frac{\Gamma(a+1)}{\Gamma(a)} nf_2 g_1 g_3 \times \{\Gamma(3m-4n+1)\Gamma(n)\} - \frac{\Gamma(2a+1)}{\Gamma(2a)} ng_1^{2a} g_2 f_3 \times \{\Gamma(3m-4n+1)\Gamma(n)\} - (2m-n)f_2 g_1^a g_3 \times \{\Gamma(m-2n+1)\Gamma(2m-n)\} - \frac{\Gamma(a+1)}{\Gamma(a)} mf_2 g_2^2 g_1^{a-1} \times \{\Gamma(2m-3n+1)\Gamma(m)\} - mf_3 g_1^{2a} g_2 \times \{\Gamma(-3n+2m+1)\Gamma(m)\} - \frac{\Gamma(a+1)}{\Gamma(a-1)} nf_2 g_2^2 g_1^{a-1} \times \{\Gamma(3m-4n+1)\Gamma(n)\} \Big]. \quad (36)$$

The above equation is the same as Eq. (12) in Section 2.1.1.

Appendix 2

From Eq. (20) given in the analysis in the Appendix 1, we equate some of the 'σ^{ua}' terms to 0 as was done for the σ^a, σ^{2a} and σ^{3a} terms.

$$0 = jf_1 g_d \frac{\Gamma(-n+1)\Gamma(j)}{\Gamma(-n+j)} t^{-n+j} + ig_{d-1} f_2 g_1^a \times \frac{\Gamma(m-2n+1)\Gamma(i)}{\Gamma(m-2n+i+1)} t^{m-2n+i} + k \frac{\Gamma(a+1)}{\Gamma(a)} f_2 g_2 g_{d-2} g_1^{a-1} \times \frac{\Gamma(2m-3n+1)\Gamma(k)}{\Gamma(2m-3n+k+1)} t^{2m-3n+k} + n \frac{\Gamma(a+1)}{\Gamma(a)} f_2 g_{d-1} g_1^a \times \frac{\Gamma(m-3n+i+1)\Gamma(n)}{\Gamma(m-2n+i+1)} t^{m-2n+i} + nf_2 g_2 g_{d-2} g_1^{a-1} \times \frac{\Gamma(2m-4n+k+1)\Gamma(n)}{\Gamma(2m-3n+k+1)} t^{2m-3n+k} + kf_3 g_1^{2a} g_{d-2} \times \frac{\Gamma(2m-3n+1)\Gamma(k)}{\Gamma(2m-3n+k+1)} t^{2m-3n+k} + n \frac{\Gamma(2a+1)}{\Gamma(2a)} f_3 g_{d-2} g_1^{2a} \times \frac{\Gamma(2m-4n+k+1)\Gamma(n)}{\Gamma(2m-3n+k+1)} t^{2m-3n+k} + nf_d g_1^{ua+1} \times \frac{\Gamma(uan-h+1)\Gamma(n)}{\Gamma((ua+1)n-h+1)} t^{(ua+1)n-h} + \cdots. \quad (37)$$

For Eq. (37) to be true, there are two solutions:

(1) $f_1, f_2, f_3, f_d, g_1, g_d, g_{d-1}, g_{d-2}, m, k, i, j$ or n are zero or
(2) powers of time of all the 8 terms in the above equation are the same, i.e. $-n+j = m-2n+i = 2m-3n+k = (ua+1)n-h$.

Only the second solution is of interest since the first solution eliminates all time dependences, reducing the viscoelastic material to an elastic one. Therefore, for Eq. (37) to be true, we need to have powers of all the time terms in it to be the same.

So, $m-2n+i = uan+n-h$.

Thus,

$$h = (ua+3)n - m - i \quad (38)$$

and $m-2n+i = -n+j$.

So $j = m—n+i$

Therefore,

$$j - i = m - n. \quad (39)$$

Then Eq. (37) can be written as follows:

$$0 = \Bigg\{ jf_1 g_d \frac{\Gamma(-n+1)\Gamma(j)}{\Gamma(-n+j)} + ig_{d-1} f_2 g_1^a \frac{\Gamma(m-2n+1)\Gamma(i)}{\Gamma(m-2n+i+1)} + k \frac{\Gamma(a+1)}{\Gamma(a)} f_2 g_2 g_{d-2} g_1^{a-1} \frac{\Gamma(2m-3n+1)\Gamma(k)}{\Gamma(2m-3n+k+1)} + n \frac{\Gamma(a+1)}{\Gamma(a)} f_2 g_{d-1} g_1^a \frac{\Gamma(m-3n+i+1)\Gamma(n)}{\Gamma(m-2n+i+1)} + nf_2 g_2 g_{d-2} g_1^{a-1} \frac{\Gamma(2m-4n+k+1)\Gamma(n)}{\Gamma(2m-3n+k+1)} + kf_3 g_1^{2a} g_{d-2} \frac{\Gamma(2m-3n+1)\Gamma(k)}{\Gamma(2m-3n+k+1)} + n \frac{\Gamma(2a+1)}{\Gamma(2a)} f_3 g_{d-2} g_1^{2a} \frac{\Gamma(2m-4n+k+1)\Gamma(n)}{\Gamma(2m-3n+k+1)} + nf_d g_1^{ua+1} \frac{\Gamma(uan-h+1)\Gamma(n)}{\Gamma((ua+1)n-h+1)} + \cdots \Bigg\} t^{-n+j}.$$

Therefore,

$$
\begin{aligned}
0 = {} & jf_1 g_d \frac{\Gamma(-n+1)\Gamma(j)}{\Gamma(-n+j)} + ig_{d-1} f_2 g_1^a \frac{\Gamma(m-2n+1)\Gamma(i)}{\Gamma(m-2n+i+1)} \\
& + k\frac{\Gamma(a+1)}{\Gamma(a)} f_2 g_2 g_{d-2} g_1^{a-1} \frac{\Gamma(2m-3n+1)\Gamma(k)}{\Gamma(2m-3n+k+1)} \\
& + n\frac{\Gamma(a+1)}{\Gamma(a)} f_2 g_{d-1} g_1^a \frac{\Gamma(m-3n+i+1)\Gamma(n)}{\Gamma(m-2n+i+1)} \\
& + nf_2 g_2 g_{d-2} g_1^{a-1} \frac{\Gamma(2m-4n+k+1)\Gamma(n)}{\Gamma(2m-3n+k+1)} \\
& + kf_3 g_1^{2a} g_{d-2} \frac{\Gamma(2m-3n+1)\Gamma(k)}{\Gamma(2m-3n+k+1)} \\
& + n\frac{\Gamma(2a+1)}{\Gamma(2a)} f_3 g_{d-2} g_1^{2a} \frac{\Gamma(2m-4n+k+1)\Gamma(n)}{\Gamma(2m-3n+k+1)} \\
& + nf_d g_1^{ua+1} \frac{\Gamma(uan-h+1)\Gamma(n)}{\Gamma((ua+1)n-h+1)} + \cdots .
\end{aligned}
$$

This procedure does not, however, give the relationship between the f and the g coefficients. That relationship is developed for each order of the analysis, and it becomes more complex as the order increases. Explicit forms are developed up to fourth order in the present work.

General interrelations are expressed as follows. From Eqs. (25), (29), (30) (34) and (35) in Appendix 1 and Eqs. (38) and (39) in Appendix 2 and if $y = m-n$, then

$$
\begin{aligned}
J(t,\sigma) = {} & g_1 t^n + g_2\sigma^a t^m + g_3\sigma^{2a} t^{m+y} + g_4\sigma^{3a} t^{m+2y} \\
& + \cdots + g_{d-1}\sigma(u-1)^a t^i + g_d \sigma u^a t^{i+y}
\end{aligned}
$$

and

$$
\begin{aligned}
E(t,\varepsilon) = {} & f_1 t^{-n} + f_2\varepsilon(t)^a t^{-[(2+a)n-m]} \\
& + f_3\varepsilon(t)^{2a} t^{-[(2a+3)n-2m]} \\
& + f_4\varepsilon(t)^{3a} t^{-[(3a+4)n-3m]} \\
& + \cdots + f_d\varepsilon(t)^{ua} t^{-[(ua+3)n-m-i]}. \qquad (40)
\end{aligned}
$$

The creep compliance can also be written as

$$
\begin{aligned}
J(t,\sigma) = {} & g_1 t^n + g_2\sigma^a t^m + g_3\sigma^{2a} t^{m+y} + g_4\sigma^{3a} t^{m+2y} \\
& + \cdots + g_{d-1}\sigma(u-1)^a t^{m+(d-3)y} + g_d\sigma^{ua} t^{m+(d-2)y}. \qquad (41)
\end{aligned}
$$

The power coefficients of time t presented in Eq. (6) were presented as independent quantities. The relations among them shown here are a consequence of the interrelation analysis.

Eq. (41) can also be written in summation notation as follows:

$$
J(t,\sigma) = \sum_{J=-1}^{X-2}\sum_{u=0}^{X-1}\sum_{d=1}^{X} g_d\sigma^{ua} t^{m+Jy}, \qquad (42)
$$

where X is any integer starting from 1.

This is the same as Eq. (12) shown in Section 2.1.2.

Correspondingly, the relaxation modulus in summation notation can be written as follows:

From Eq. (42), the power of time of the dth term in creep is $m+Jy$.

From Eq. (38), the derived power of time for the dth term in relaxation is $(ua+3)n-m-i$, where i is the power of time of the $(d-1)$th term in creep, which is also $m+(J-1)y$, so $i = m+(J-1)y$ can be substituted in $(ua+3)n-m-i$.

Solving that gives us the power of time of the dth term in relaxation as $(ua+3)n-2m-(J-1)y$, so Eq. (40) for relaxation modulus in summation notation becomes

$$
E(t,\varepsilon(t)) = \sum_{J=-1}^{X-2}\sum_{u=0}^{X-1}\sum_{d=1}^{X} f_d\varepsilon(t)^{ua} t^{-[(ua+3)n-2m-(J-1)y]}, \qquad (43)
$$

where X is any integer starting from 1.

This is the same as Eq. (13) shown in Section 2.1.2.

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