FINITE ELEMENT ANALYSIS OF STRESS CONCENTRATION AROUND A BLUNT CRACK IN A COSSERAT ELASTIC SOLID

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Abstract

The problem of a blunt edge notch of elliptic contour in a strip of Cosserat (micropolar) elastic material under tension has been analyzed via two-dimensional (plane stress) finite element analysis. Both 3-node constant strain triangular elements and 4-node isoparametric elements were used. Three cases were explored: Cosserat characteristic length much less than the crack root radius, equal to the crack root radius; and comparable to the crack length. Stress concentration factors were found to be reduced by as much as a factor of 2.6 in comparison with a classically elastic material. For the special case of a classically elastic material, the finite element results agreed well with the classical analytic solution.

Introduction

In composite materials with stress concentrations due to holes or cracks, the observed fracture behavior is not correctly predicted by the classical theory of anisotropic elasticity [1-3]. The experimental stress concentrations are consistently less than the theoretical ones. Such discrepancies may in part be accounted for by Weibull analysis of flaw populations; nevertheless, the discrepancies persist even under load conditions in which the elastic limit of the material is not approached. Specifically, strain distributions have been observed which differ from the predictions of classical elasticity, particularly for small holes and small cracks [4]. Observed concentrations of strain are less than predicted values. Strain fields around large holes, by contrast, follow classical predictions [5]. Such phenomena are attributed to the composite microstructure, however a rational method for predicting stress concentrations in composites is not yet available.

Generalized continuum theories have been advanced, which incorporate some of the microstructural degrees of freedom of the composite. Specifically, the Cosserat theory of elasticity [6], also known as micropolar elasticity [7] incorporates a local rotation of points as well as the translation assumed in classical elasticity. In the isotropic Cosserat solid there are six elastic constants as described below, in contrast to the classical solid in which there are two. Certain combinations of Cosserat elastic constants have dimensions of length and are referred to as characteristic lengths. In higher order composite models of structured media it has been predicted that the characteristic lengths are of the order of magnitude of the size of the structural elements [8,9]. Recent experimental studies of natural fibrous solids [10] and of polymer foams [11] support this general view. Cosserat elasticity and related theories are thought to offer advantages over classical elasticity in the prediction of stresses in materials with microstructure. In particular, analytical solutions for stress concentration around circular holes [12,13] and elliptic holes [14,15] in plates disclose smaller stress concentration factors in a Cosserat solid as opposed to a classical solid. Similar results have been obtained for predicted stress-intensity factors for cracks [16] in plates as well as planar circular cracks [17] in three-dimensional solids. The deviations are most significant when the hole size or crack length is no more than ten times the Cosserat characteristic length.

The case of a 'blunt' notch of nonzero radius of curvature at the notch tip has not to the authors' knowledge, been treated in the literature. This case is important in that (i) classical theories fail to make correct predictions suitable for design purposes, and (ii) crack tip radii may be comparable to the dimensions of the microstructure. It is in such situations that generalized continuum theories are likely to be most useful. Although analytical solutions for the elliptic hole problem might be used in this regard, published results have dealt with ellipses of small eccentricity which do not approximate notches very well. Extension of such solutions to cases of large eccentricity appear to be problematical numerically and operationally. Consequently, a finite element method developed by the authors [18] and its enhanced version [19] were applied to this problem.

Cosserat elastic solids: constitutive equations

The constitutive equations for a linear isotropic Cosserat (micropolar) elastic solid are [7]:

$$t_{kl} = \lambda e_{rr}\, \delta_{kl} + (2\mu + \kappa) e_{kl} + \kappa e_{klm}(r_m - \varphi_m) \qquad (1)$$

$$m_{kl} = \alpha\, \varphi_{r,r}\, \delta_{kl} + \beta \varphi_{k,l} + \gamma \varphi_{l,k} \qquad (2)$$

in which t_{kl} is the asymmetric force stress, m_{kl} is the couple stress, $e_{kl} = (u_{k,l} + u_{l,k})/2$ is the small strain, u is the displacement, δ_{kl} is the Kronecker delta, and e_{klm} is the permutation symbol. The usual Einstein summation convention for repeated subscripts is used in this article; the commas denote differentiation with respect to spatial coordinates. The microrotation ϕ in Cosserat elasticity is kinematically distinct from the macrorotation $r_k = (e_{klm}u_{m,l})/2$. In three dimensions, the isotropic Cosserat elastic solid requires six elastic constants λ, μ, α, β, γ, and κ for its description. When α, β, γ, κ vanish the solid becomes classically elastic. The following combinations of elastic constants are conceptually useful as so-called technical elastic constants. These are [13,20]: Young's modulus $E = (2\mu+\kappa)(3\lambda+2\mu+\kappa)/(2\lambda+2\mu+\kappa)$, shear modulus $G = (2\mu+\kappa)/2$, Poisson's ratio $\nu = \lambda/(2\lambda+2\mu+\kappa)$, characteristic length for torsion $l_t = [(\beta+\gamma)/(2\mu+\kappa)]^{1/2}$, characteristic length for bending $l_b = [\gamma/2(2\mu+\kappa)]^{1/2}$, coupling number $N = [\kappa/2(\mu+\kappa)]^{1/2}$, and polar ratio $\Psi = (\beta+\gamma)/(\alpha+\beta+\gamma)$

When the above constitutive equations are specialized to two dimensions, under conditions of plane stress and plane couple stress, there is only one component of the microrotation vector, which is perpendicular to the plate. Gradients of this rotation are coupled to permissible components of couple stress by γ only; β is not involved. Consequently, in two dimensions, four elastic constants are required for the isotropic Cosserat solid: E, ν, l_b, and N, compared with two, E, ν in the classical case. For a two dimensional orthotropic material, four elastic constants are needed in the classical case and nine in the Cosserat elastic case.

Finite element model

The finite element model is based on a variational formulation presented earlier. The basis for this formulation is the total potential energy P for a micropolar medium [18]: The finite element method used for the present study is based on the same variational principle developed in Ref.[18].

Two finite element approaches for plane stress orthotropic Cosserat elasticity have been developed based on the above energy formulation. The method presented earlier [18] used triangular constant strain elements, while the second method uses 4 node isoparametric elements [19]. The numerical integration scheme used for the isoparametric element was Gauss-Legendre quadrature [21]. In the present work, the Skyline technique [22] was used to solve the simultaneous linear algebraic equations. Computation time improved from 2 (CPU) hours to 3 (CPU) minutes in a VAX 11/780 system.

Finite element model

A strip of material 18 mm wide and 25.4 mm long, with a single edge notch of elliptic contour, subjected to uniaxial tension, was considered in this study. Fig. 1 displays the mesh for the 4-node isoparametric elements. Owing to the single plane of symmetry in this problem, only half the strip was discretized in these models. The same number of elements was used regrdless of notch depth, except in convergence studies in which the number of elements was deliberately varied. There were 1563 3-node elements and 640 4-node elements. To achieve uniaxial tension, a force traction was distributed uniformly across the 18 mm width and all other boundaries were assumed to be free of applied tractions. The notch tip radius of curvature was kept constant at 0.5 mm and various values of notch length and material elastic constants were assumed. In two dimensions,the following four elastic constants are required to describe an isotropic Cosserat solid: E, ν, l_b, and N. In the evaluation of stress concentration factors, Young's modulus E has no influence. Poisson's ratio n was assumed to be 0.3. Regarding the characteristic length l_b, three situations were distinguished: Cosserat characteristic length much less than the crack tip radius of curvature, characteristic length comparable to the crack tip radius of curvature, and the characteristic length comparable to the crack length. Stress concentration factors were calculated under the above assumptions. In addition, the distribution of stress in the vicinity of the notch tip was calculated for a particular case. Each calculation, for a particular set of elastic constants for a given notch length, required approximately 46 seconds on a VAX 11/750 or on a PRIME 750.

The validity of the finite element models was tested by the following procedures: (i) In the classical case, finite element results were compared with the predictions of the classical Inglis formula for an elliptic hole in an infinite sheet under tension, (ii) In the Cosserat elastic case, finite element results were compared with the predictions of published analytical solutions for an elliptic hole, (iii) In both cases, finite element results for 3-node and 4-node elements were compared, and (iv) Convergence studies were performed in which the number of elements was varied until the predicted stress concentration factor converged to a constant value.

Results

Results of the analysis of isotropic solids with 4-node elements are displayed in Figures 2-4. In Fig. 2, we consider first the case N = 0 which corresponds to classical elasticity as in [13,18]. This case compares favorably with the classical analytical Inglis formula [23] for an elliptic hole in an infinite plate. This equation states that the stress concentration factor is: $scf = 1 + 2\sqrt{a/r}$ in which a is the ellipse semimajor axis and r is the radius of curvature. In Fig. 2, the Inglis formula has been corrected [24] for the finite width of the strip; the correction is claimed to be valid for cracks less than 0.6 of the strip width. As indicated in Table 1, the finite width corrections become important when the notch depth becomes an appreciable fraction of the strip width. Agreement between the 4-node finite element results and the corrected Inglis formula was within 0.5%, 1.5%, and 0.3% for the three notch lengths displayed in Table 1.

In Fig. 2, the characteristic length is 0.1 mm, less than the notch root radius r which was assumed to be 0.5 mm throughout. For Cosserat elastic solids in which l = 0.1 mm and N > 0, modest reductions in the stress concentration factor, up to 12%, are predicted, with larger reductions for larger values of N, up to N = 0.93. The permissible range of values[13] for N is $0 \le N \le 1$. However, in the present displacement type finite element analysis a numerical instability begins to occur for N > 0.95 since the stiffness matrix becomes singular for N = 1. A restricted range of values of N, from 0 to 0.93, was therefore used in the computations. Table 1 also displays a comparison of results obtained via 3-node and 4-node finite elements. The 4-node elements proved to be more accurate in correctly predicting the classical stress concentrations. The difference between the 4-node and 3-node results was 4% for short notches and 12% for long ones.

When the Cosserat characteristic length is 1 mm, or twice the notch root radius, as shown in Fig. 3, larger reductions in the stress concentration factor, up to a factor of 1.5, are predicted. When the characteristic length is 10 mm and exceeds the notch length, as shown in Fig. 4, the stress concentration factor is reduced by a factor of 1.5 for short cracks and up to 2.39 for long ones. Very large values of the characteristic length, e.g. 100 mm or 10 m, produced little additional change in the predicted stress concentration factor, 0.1% for short notches and 2.4% for long ones, in comparison with the values for l = 10 mm.

Several convergence studies were performed to verify the reliability of the finite element meshes used. Results of convergence studies for N = 0 are shown in Table 2. In these studies, the number of elements was varied to ascertain whether the solution would converge to a constant value. In the case of the 3-node elements, convergence was not complete even with 1562 elements, while for the 4-node elements, 640 elements sufficed for convergence. Nevertheless, for 1562 3-node elements used for the analysis of a relatively short notch of length c (c/r = 2.5 in Table 2), the predicted stress concentration factor was within 2.7% of the value given by the corrected Inglis formula.

The distribution of stress in the vicinity of the notch tip was computed for the case l = 1 mm, various N, and a notch of depth c = 5 mm and a root radius of r = 0.5 mm. The results are shown in Figure 5. The influence of N, hence the influence of assumed Cosserat elastic behavior, is confined to a small region around the notch tip. The maximum stress is significantly reduced, but the stresses some distance, e.g. $y' = y - c > r$, from the notch tip are not affected much. Moreover, in both the classical case and the Cosserat elastic case, the stress distribution is given by $s = (\text{const})/\sqrt{y'}$, except in the immediate vicinity of the notch tip.

Discussion

Several comparisons can be made between the present finite element results and earlier analytical results for Cosserat solids in similar loading geometries. Kim and Eringen [14] performed an analytical study of an ***elliptic*** hole of small eccentricity in a micropolar plate under tension. They found the classical stress concentration for this ellipse to be 3.65, for n = 0.25. For a sufficiently large value of characteristic length greater than the hole size, and for N = 0.5, the stress concentration became 2.90, a reduction of a factor of 1.26. In the present study of an elliptic edge notch of comparable eccentricity we find (Fig. 4) the stress concentration is reduced by a factor of 1.24 for C = 0.8 mm, l = 10 mm, N = 0.5, and n = 0.30. This may be regarded as satisfactory agreement.

Results for dissimilar geometries may also be compared. In this article, three lengths are considered: the crack length, the crack root radius, and the Cosserat characteristic length of the material. In the case of a ***sharp*** crack, the root radius is zero and only two lengths remain. This situation was examined analytically by Sternberg and Muki [16] who found that for n = 0.25, the stress intensity factor for a sharp center crack was reduced by 26% as l increased from zero to values greater than the crack length. This analysis was for a couple stress elastic solid, equivalent to
N = 1 in the present context [13]. The present results, by contrast, predict that much larger reductions in the stress concentration factor occur in a ***blunt*** notch in which the Cosserat characteristic length exceeds the notch root radius. As for the stress distribution, a $1/\sqrt{y'}$ distribution was obtained both in [16] and in the present results. For a ***sharp*** notch, the stress becomes singular at the notch tip, while for a blunt notch, the stress remains finite at the notch tip. It is interesting to note that the order of the singularity in the sharp notch problem is preserved in the departure from classical elasticity theory [16].

In this article, N = 0 has been considered to be equivalent to the case of classical elasticity. The assumption of such equivalence is not, however, always warranted [25]. The use of N = 0 for classical elasticity in this article is justified by the fact that self-equilibrated distributions of surface traction and couple traction considered in [25] were not applied in this study. Furthermore the finite element results for N = 0 were found to agree with the analytical predictions of classical elasticity.

Implications of the present study may be considered in relation to the fracture mechanics of structured materials, particularly composites. Such materials are regarded as equivalently homogeneous elastic media for the purpose of calculating macroscopic stresses. Such equivalence is warrantable only if the characteristic distance over which stresses vary is sufficiently large compared with the structure size. In the case of notches, predictions based on classical elasticity are very much in error. Empirical, modified fracture criteria have been proposed to deal with these discrepancies [1-3; 26]. These are, however, ad hoc and there is no guarantee that they will apply to notches of different geometry. Cosserat elasticity, by contrast, incorporates some of the degrees of freedom of the structure; the new elastic constants can be predicted from theoretical consideration of the structure or can be obtained from experiment.

Conclusions

1. In finite element analysis of Cosserat elastic solids involving large strain gradients, 4-node isoparametric elements are superior to 3-node constant stress elements. This observation confirms the expectation that 4-node elements in general offer superior performance.

2. Stress concentrations around an elliptic notch in a Cosserat elastic solid are reduced in comparison with classical values.

3. Significant reductions, by more than a factor of two, in stress concentration can occur when the Cosserat characteristic length is of the order of the notch root radius. This reduction is much more pronounced for a blunt notch than for the sharp crack considered in an earlier study.

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TABLE 1 Prediction of stress concentration factor (SCF)

		l = 0.1 mm		l = 1.0 mm	
c/r		N = 0.	N = 0.93	N = 0.	N = 0.93
1.00	4-node SCF	3.0038	2.7112	3.0038	2.0428
	3-node SCF	2.8796	2.6076	2.8796	1.9803
	Inglis(1) SCF	3.0000	NA	3.0000	NA
	Inglis(2) SCF	2.9884	NA	2.9884	NA
2.50	4-node SCF	4.3093	3.9194	4.3093	2.7934
	3-node SCF	4.1324	3.8054	4.1324	2.7492
	Inglis(1) SCF	4.1623	NA	4.1623	NA
	Inglis(2) SCF	4.2457	NA	4.2457	NA
5.00	4-node SCF	6.0314	5.4736	6.0314	3.9231
	3-node SCF	5.6875	5.2733	5.6875	3.8654
	Inglis(1) SCF	5.4721	NA	5.4721	NA
	Inglis(2) SCF	6.0487	NA	6.0487	NA

(1): Classically elastic Inglis formula for an elliptic hole in an infinite region.
(2): Inglis formula for an elliptic notch, corrected for finite strip width.
3-node: Finite element model with 1562 3-node constant stress elements.
4-node: Finite element model with 640 4-node isoparametric elements.

TABLE 2 Convergence study, stress concentration factor (SCF)

4-node elements, c/r=10.

Elements	496	532	568	604	640	676	712
Nodes	542	580	618	656	694	732	770
SCF, N = 0.00	9.2481	9.3661	9.4391	9.4769	9.4880	9.4802	9.4595

3-node elements, c/r=2.5.

Elements	986	1238	1346	1418	1490	1562
Nodes	539	672	729	767	805	843
SCF, N = 0.00	3.2581	3.7849	3.9361	4.0158	4.0807	4.1324

Figure captions

Fig. 1. Finite element mesh. There are 640 4-node isoparametric elements. Arrows indicate the applied tension.

Fig. 2. Stress concentration factor vs crack length, for an isotropic plate under tension.
˙ - Finite element prediction, l = 0.1mm, various N. The case N = 0 corresponds to classical elasticity.

Δ: Inglis formula of classical elasticity, corrected for the finite width of the strip.

Fig. 3. Stress concentration factor vs crack length, for an isotropic plate under tension.

˙ - Finite element prediction, l = 1 mm, various N.

Fig. 4. Stress concentration factor vs crack length, for an isotropic plate under tension.

˙ - Finite element prediction, l =10 mm, various N.

Fig. 5. Stress distribution. $s_{xx}(y)/s_0$ vs y for l = 1 mm and various N. s_0 is the applied stress.

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